Feagin's Order 10, 12, and 14 Methods

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DifferentialEquations.jl includes Feagin's explicit Runge-Kutta methods of orders 10/8, 12/10, and 14/12. These methods have such high order that it's pretty much required that one uses numbers with more precision than Float64. As a prerequisite reference on how to use arbitrary number systems (including higher precision) in the numerical solvers, please see the Solving Equations in With Chosen Number Types notebook.

0.1 Investigation of the Method's Error

We can use Feagin's order 16 method as follows. Let's use a two-dimensional linear ODE. Like in the Solving Equations in With Chosen Number Types notebook, we change the initial condition to BigFloats to tell the solver to use BigFloat types.

```
using DifferentialEquations
const linear_bigα = big(1.01)
f(u,p,t) = (linear_bigα*u)

# Add analytical solution so that errors are checked
f_analytic(u0,p,t) = u0*exp(linear_bigα*t)
ff = ODEFunction(f,analytic=f_analytic)
prob = ODEProblem(ff,big(0.5),(0.0,1.0))
sol = solve(prob,Feagin14(),dt=1//16,adaptive=false);
```

```
retcode: Success
Interpolation: 3rd order Hermite
t: 17-element Array{Float64,1}:
 0.0
 0.0625
 0.125
 0.1875
 0.25
 0.3125
 0.375
 0.4375
 0.5
 0.5625
 0.625
 0.6875
 0.75
 0.8125
 0.875
 0.9375
 1.0
```

```
u: 17-element Array{BigFloat,1}:
 0.50
 0.5325799879535391294151756922663100127575701279367674657299889518448727031384926
 0.5672828871371837684101760938113319028166849848855565501534311815341659662084392
 0.6042470263955404578588403482610764707515845791230606683491210486718827818716904
 0.643619748077397554810695151333134437286938781958375274884622802079377793549762
 0.685557995355440557982096349246370736930808950070979750802507189818563351642024
 0.7302289378157059336084349405886010308809987247048142151879999305882306169718902
 0.777810637810428680311119693945339493753691675945670302006794689942184819812709
 0.8284927602304253869042839670047645226327964053287150891084535034410969675378266
 0.8824773285262286697576984867723954313541934495294660822778536314177585961942936
 0.9399795299915405153402248568618947272920440084137115643604676307008419853257788
 1.001228573518936040932179654581424903118245024943528049619445960282439336451516
 1.06646860324690824654467242384670608266022713505450493171046906577618017533485
 1.135959671740132190447259472795962527154069506739201242159983984002811333382739
 1.209978776582131731617034761065124311009358614495571372335128166979326800445978
 1.288820964512299462579706253102020998553494397009716401282671347081826473196884
 1.372800507508458259187784036450083092262707343306282981254591926965493411372691
```

```
println(sol.errors)
```

```
Dict{Symbol,BigFloat}(:l∞ => 2.1975104034266099178147026326495605606836593676837803246358016102973498729096 55e-23,:final => 2.19751040342660991781470263264956056068365936768378032463580161029734987290965 5e-23,:l2 => 1.06150159781476863589451467759071276224836468652759635990282684174054997568816 1e-23
)
```

Compare that to machine ϵ for Float64:

```
eps(Float64)
```

```
2.220446049250313e-16
```

The error for Feagin's method when the stepsize is 1/16 is 8 orders of magnitude below machine ϵ! However, that is dependent on the stepsize. If we instead use adaptive timestepping with the default tolerances, we get

```
sol =solve(prob,Feagin14());
println(sol.errors); print("The length was $(length(sol))")
```

```
Dict{Symbol,BigFloat}(:l∞ => 1.545738883943140962546537598609759219816414790728029220638828884206395861982752e-09,:final => 1.545738883943140962546537598609759219816414790728029220638828884206395861982752e-09,:l2 => 8.92506687020233040992442119216219346250638833226107472510994921806776340513799 3e-10
```

```
)
The length was 3
```

Notice that when the stepsize is much higher, the error goes up quickly as well. These super high order methods are best when used to gain really accurate approximations (using still modest timesteps). Some examples of where such precision is necessary is astrodynamics where the many-body problem is highly chaotic and thus sensitive to small errors.

0.2 Convergence Test

The Order 14 method is awesome, but we need to make sure it's really that awesome. The following convergence test is used in the package tests in order to make sure the implementation is correct. Note that all methods have such tests in place.

```
using DiffEqDevTools
dts = 1.0 ./ 2.0 .^(10:-1:4)
sim = test_convergence(dts,prob,Feagin14())
```

```
DiffEqDevTools.ConvergenceSimulation{DiffEqBase.ODESolution{BigFloat,1,Arra
y{BigFloat,1},Array{BigFloat,1},Dict{Symbol,BigFloat},Array{Float64,1},Arra
y{Array{BigFloat,1},1},DiffEqBase.ODEProblem{BigFloat,Tuple{Float64,Float64
},false,DiffEqBase.NullParameters,DiffEqBase.ODEFunction{false,typeof(Main.
##WeaveSandBox#256.f),LinearAlgebra.UniformScaling{Bool},typeof(Main.##Weav
eSandBox#256.f_analytic),Nothing,Nothing,Nothing,Nothing,Nothing,Nothing,No
thing,Nothing,Nothing,Nothing,Nothing},Base.Iterators.Pairs{Union{},Union{}
,Tuple{},NamedTuple{(),Tuple{}}},DiffEqBase.StandardODEProblem},OrdinaryDif
fEq.Feagin14,OrdinaryDiffEq.InterpolationData{DiffEqBase.ODEFunction{false,
typeof(Main.##WeaveSandBox#256.f),LinearAlgebra.UniformScaling{Bool},typeof
(Main.##WeaveSandBox#256.f_analytic),Nothing,Nothing,Nothing,Nothing,Nothin
g,Nothing,Nothing,Nothing,Nothing,Nothing,Nothing},Array{BigFloat,1},Array{
Float64,1},Array{Array{BigFloat,1},1},OrdinaryDiffEq.Feagin14ConstantCache{
BigFloat,Float64}},DiffEqBase.DEStats}}(DiffEqBase.ODESolution{BigFloat,1,A
rray{BigFloat,1},Array{BigFloat,1},Dict{Symbol,BigFloat},Array{Float64,1},A
rray{Array{BigFloat,1},1},DiffEqBase.ODEProblem{BigFloat,Tuple{Float64,Floa
t64},false,DiffEqBase.NullParameters,DiffEqBase.ODEFunction{false,typeof(Ma
in.##WeaveSandBox#256.f),LinearAlgebra.UniformScaling{Bool},typeof(Main.##W
eaveSandBox#256.f_analytic),Nothing,Nothing,Nothing,Nothing,Nothing,Nothing
,Nothing,Nothing,Nothing,Nothing,Nothing},Base.Iterators.Pairs{Union{},Unio
n{},Tuple{},NamedTuple{(),Tuple{}}},DiffEqBase.StandardODEProblem},Ordinary
DiffEq.Feagin14,OrdinaryDiffEq.InterpolationData{DiffEqBase.ODEFunction{fal
se,typeof(Main.##WeaveSandBox#256.f),LinearAlgebra.UniformScaling{Bool},typ
eof(Main.##WeaveSandBox#256.f_analytic),Nothing,Nothing,Nothing,Nothing,Not
hing,Nothing,Nothing,Nothing,Nothing,Nothing,Nothing},Array{BigFloat,1},Arr
ay{Float64,1},Array{Array{BigFloat,1},1},OrdinaryDiffEq.Feagin14ConstantCac
he{BigFloat,Float64}},DiffEqBase.DEStats}[retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.0009765625, 0.001953125, 0.0029296875, 0.00390625, 0.0048828125,
 0.005859375, 0.0068359375, 0.0078125, 0.0087890625  ...  0.9912109375, 0.992
1875, 0.9931640625, 0.994140625, 0.9951171875, 0.99609375, 0.9970703125, 0.
998046875, 0.9990234375, 1.0]
u: BigFloat[0.50, 0.500493407353274144224016740778348649218060302161584129
4522027941156602112195 99, 0.500987301608180818440355281223188950584910597683
376621349294923926084136802 6092, 0.501481683245201714271982570998354212145 3
333631661309529343897218820605529258 6, 0.501976552745292669741386303121966 4
328338088601822445017121888607974310367490053, 0.502471910589884137165215 01
795632390021433166490802070985470711669875234308168 98, 0.502967757260881651
5106435191675246046427648760687772460544794831604624949834855, 0.5034640932
```

```
406662992139192389830628016370303392435659971381052766108783597241391, 0.50
396091901209518746154563970525241120918072457951989067083504574325947816539
15, 0.504458235058501913934559188511648417802868424125759953366000451921798
6099443769  ...  1.3606681506044349509999505354431852352163391999270684367421
06414044679239515405, 1.362010877946183268827001169225864534157243515073039
951153177861472631409617242, 1.36335493031101930838372463784595682752195537
609659291992754149268935330476256, 1.36470030900649533914994667058892723740
3312989133498476567243275650792209901316, 1.3660470153414539424145962795447
72708050551524408051147344999020270553111340215, 1.367395050626029284574375
015655225687307700620914621502414819287823171439227302, 1.36874441617164839
1688936077411190219324498098498429005876504282210656625474981, 1.3700951132
91032425293813214601928520794629811024108089775792586183724683573052, 1.371
447143298197959472340593110913203812626370594321605117950295569839313271375
, 1.37280050750845825918780601155411735836188549033294491231479506286094526
5969766], retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.001953125, 0.00390625, 0.005859375, 0.0078125, 0.009765625, 0.01
171875, 0.013671875, 0.015625, 0.017578125  ...  0.982421875, 0.984375, 0.986
328125, 0.98828125, 0.990234375, 0.9921875, 0.994140625, 0.99609375, 0.9980
46875, 1.0]
u: BigFloat[0.50, 0.5009873016081808184403552812231889505849105976869970947
216582055241988418330043, 0.50197655274529266974138630312196643283380886018
94997464536470643897825028575861, 0.502967757260881651510643519167524604642
764876079681602511069749881119893067103, 0.50396091901209518746154563970525
24112091807245940877416493974506008240426203329, 0.504956041863697037016356
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345471746020389471304909424091056168364541340183078255553277085, 0.50695218
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507953215783031129154683858104667438830905843059054531062193869152593801987
2, 0.5089562218366775389970434828264868922469483072826175429449686273069735
423750136  ...  1.34864301545859247967509602307176367401641641210475048875695
337449929440786918, 1.35130605029464067315359841454792588096269341231012308
6924207034022962413754022, 1.3539743435678414104792857791059277027193819496
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69225864534157243520073188743602186030419166325630835, 1.364700309006495339
14994667058892723740331299415338282713705459498054912456936, 1.367395050626
0292845743750156552256873077006259542998550290571206539333933847354, 1.37009
51132910324252938132146019285207946298160836390422151163694340862252063, 1.
372800507508458259187806011554117358361885495412387216596637263939057644809
585], retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.00390625, 0.0078125, 0.01171875, 0.015625, 0.01953125, 0.0234375
, 0.02734375, 0.03125, 0.03515625  ...  0.96484375, 0.96875, 0.97265625, 0.97
65625, 0.98046875, 0.984375, 0.98828125, 0.9921875, 0.99609375, 1.0]
u: BigFloat[0.50, 0.5019765527452926697413863031219664328338089596679027919
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73910752677848686355143273445556, 0.505953129688082334547174602038947130490
9427099049813605553848314515187463009496, 0.5079532157830311291546838581046
674388309062457103293891502691351497465467261813, 0.50996120842930350923278
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837, 0.51807287149253063717412013329248710548519771111335349047947245099497
99099094199  ...  1.324910628454177546699995024759245940440919035869353532178
274979609386868635781, 1.33014813993405462867526880344471252204348922373995
5282684225096368730900386558, 1.3354063558493198164900848637876127679604780
```

```
498119585885764231213717590053539777, 1.340685358046790322335041702613370942
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51653325099084463360226629376573437098670177771688, 1.3513060502946406731535
98414547925880962760896126504201689516014513211559127835, 1.356647905661521
606100766158464652339627599813198025862866692566448305652099871, 1.36201087
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66, 1.372800507508458259187806011554117358361955140865683163444664161725836
562110807], retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.0078125, 0.015625, 0.0234375, 0.03125, 0.0390625, 0.046875, 0.05
46875, 0.0625, 0.0703125  ...  0.9296875, 0.9375, 0.9453125, 0.953125, 0.9609
375, 0.96875, 0.9765625, 0.984375, 0.9921875, 1.0]
u: BigFloat[0.50, 0.5039609190120951874615456397052524112111879130152678185
972878772909801438539725, 0.50795321578303112915468385810466743883495202114
45279229156914347577732228682629, 0.511977138882330923641985750275520143860
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3024626851557639184785285451286948438374521217071188324, 0.5242411814762751
337774973218812134360586606206771934231862500076957565363308646, 0.52839413
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2659, 0.5367990003530323275101958345252554486169547480536807778674613463641
61391270671  ...  1.278691378528666663061419567734810219225057268467188197890
749511286039814682961, 1.28882096451229946257972559447843216250288715170826
8911920429452719237031672889, 1.2990307954353467102700618873848859068511184
7286542819859803450631324042075354, 1.3093215069852207083608767825998856173
05680877664407188219768945081296125363463, 1.319693739885146473355596260460
075276820407682877165061218322278576772888141989, 1.33014813993405462867526
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22335041702613370942986414700099829148697106600907447993876335, 1.351306050
294640673153598414547925881640827460343653263042386265204459670148168, 1.36
201087794618326882700116922586453484617428790917305214715458105432837901989
4, 1.3728005075084582591878060115541173590617414861285190771349018784370260
32283726], retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.015625, 0.03125, 0.046875, 0.0625, 0.078125, 0.09375, 0.109375,
0.125, 0.140625  ...  0.859375, 0.875, 0.890625, 0.90625, 0.921875, 0.9375, 0
.953125, 0.96875, 0.984375, 1.0]
u: BigFloat[0.50, 0.5079532157830311291546838581046674228648435931525498190
278213939882467623443407, 0.51603293884864516922308309058199688204679557555
37640078345474392168602557520595, 0.524241181476275133777497321881213386612
0487057078897985952628075072800031634668, 0.5325799879535391294151762250959
425943293258372491737121300708996931105182686892, 0.54105143508537636149437
50093333620112324860397855474272871650047471573253335466, 0.549657632711281
6769034687550989237307955190439443562958216377582793355171347836, 0.5584007
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29958, 0.576306333560029601749846103643822584449515926861513757846110140307
5168130161427  ...  1.1910336808448971384801387451313626311041821234303624933
41676788830879226337858, 1.209978776582131731617051708708410892779229149607
19360099713235492900284779981, 1.229225221188223144617416241361990712052620
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0708360876782599883106227194078928307355592757175919985164258564, 1.3301481
3993405462867526880344470992985954682818207565575624581067560987566062, 1.3
513060502946406731535984145479232050719649354675944552369964535610982815942
```

```
27, 1.37280050750845825918780601155411459675721123243343289336285585803265
914211007], retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.03125, 0.0625, 0.09375, 0.125, 0.15625, 0.1875, 0.21875, 0.25, 0
.28125  ...  0.71875, 0.75, 0.78125, 0.8125, 0.84375, 0.875, 0.90625, 0.9375,
 0.96875, 1.0]
u: BigFloat[0.50, 0.516032938848645169223083090576164551016602523605979342
081272221990055728561679, 0.532579987953539129415176225083903894641931169817
6330530726315794171525645034893, 0.5496576327112816769034687550802866340567
114355433590002071694871709322410405931, 0.56728288713718376841017722888328
28270073935705216193322632159538121017711969774, 0.585473310815890461621019
6190827662219055210008829792702890722633372001092380584, 0.6042470263955404
578588421618108943195685209948140048324759492077870732303989337, 0.62362273
76428912250695033937265522498285111444862489660878884721013969830771631, 0.
643619748077397554810697726960708152129622579985972873985637448596666363180
053, 0.66425798020280820325088307482585132534106704873625388888034286900680
76152641046  ...  1.033333846504426712082580354960107935043970613756493163451
1294626091384818214, 1.0664686032469082465446852271794933182809166964255378
32438762276651139095129374, 1.100665855046623660308919551395610359208286716
684431112070414408183611765253205, 1.13595967174013219044727424686007597453
6172903481653049159364319427066929995183, 1.1723852156432053491987697297701
67158307196511889785786439894918782978322870963, 1.209978776582131731617051
708325497527117244553498434756316642412641437610370797, 1.24877780804833135
8756513829430099007097842940139504429315922973879445769691784, 1.2888209645
1229946257972559404143259768018368155787883476891071715795301279543, 1.3301
48139934054628675268802978666438630864641254421813595879534440847470751688,
 1.372800507508458259187806011057611283394590006590596578199873499201906331
611072], retcode: Success
Interpolation: 3rd order Hermite
t: [0.0, 0.0625, 0.125, 0.1875, 0.25, 0.3125, 0.375, 0.4375, 0.5, 0.5625, 0
.625, 0.6875, 0.75, 0.8125, 0.875, 0.9375, 1.0]
u: BigFloat[0.50, 0.5325799879535391294151756922663100127575701279367674657
299889518448727031384926, 0.56728288713718376841017609381133190281668498488
55565501534311815341659662084392, 0.604247026395540457858840348261076470751
5845791230606683491210486718827818716904, 0.6436197480773975548106951513331
34437286938781958375274884622802079377793549762, 0.685557995355440557982096
349246370736930808950070979750802507189818563351642024, 0.73022893781570593
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```

```
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8446539816666631293486902250519752824131061460869466635839790668e-41, 6.998
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e-37, 2.7616046742578991765834973429055265111070658679880654574554903808257
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, 3.24060760516074676637178554271828070823716806609832930426777898390162348
7961701e-41, 3.264565979149024498598621687244084221464920048688554495210368
172485686228822379e-37, 1.2947776667473864852636114197311110560713898649841
76915402703871046417929063523e-33, 2.35148503019100306142594944698233564880
1181524332244933545614443786091245762492e-28, 1.061501597814768635894514677
590712762248364686527596359902826841740549975688161e-23]), 7, Dict(:dts =>
[0.0009765625, 0.001953125, 0.00390625, 0.0078125, 0.015625, 0.03125, 0.062
5]), Dict{Any,Any}(:l∞ => 14.2933275461038524350008931328481604055650481625
4374715376150534187461411604701,:final => 14.293327546103852435000893132848
16040556504816254374715376150534187461411604701,:l2 => 14.30280974051840423
232019057634315242594313233119811212889763182960978082577156), [0.000976562
5, 0.001953125, 0.00390625, 0.0078125, 0.015625, 0.03125, 0.0625])
```
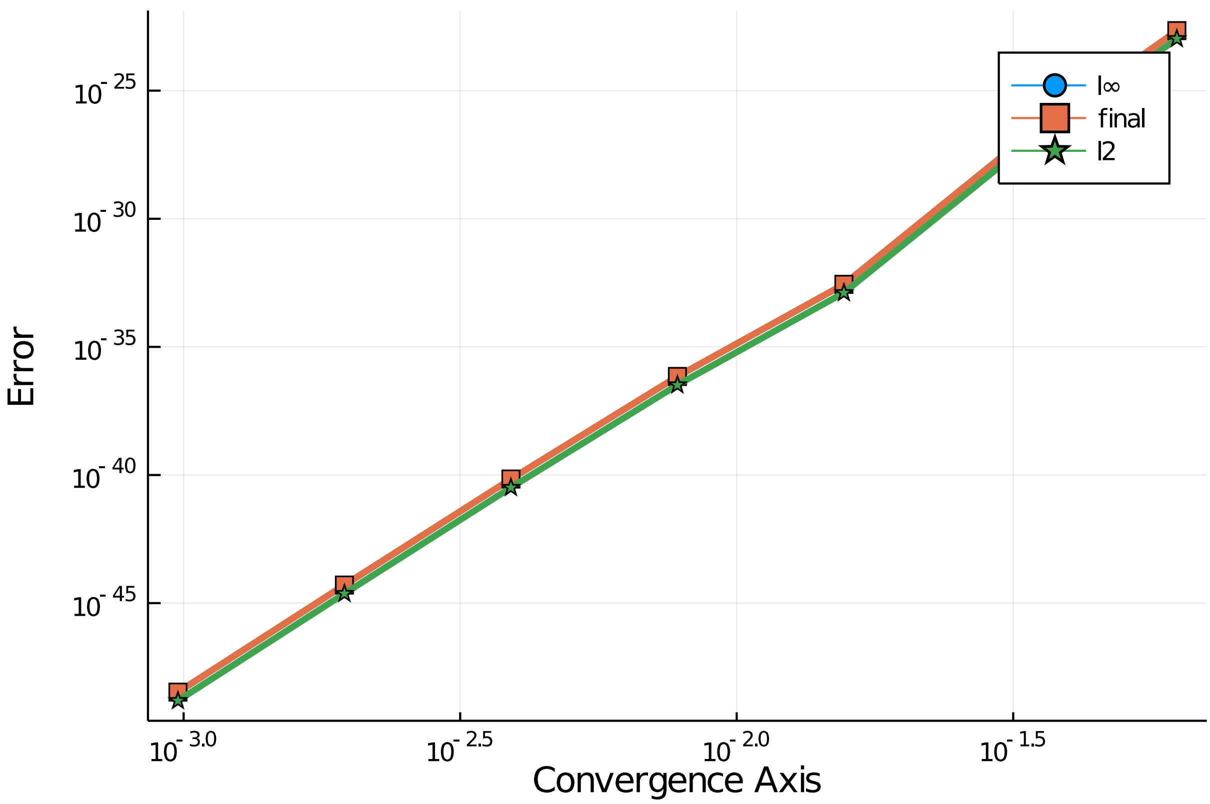

For a view of what's going on, let's plot the simulation results.

```
using Plots
gr()
plot(sim)
```



This is a clear trend indicating that the convergence is truly Order 14, which is the estimated slope.

0.3 Appendix

This tutorial is part of the DiffEqTutorials.jl repository, found at: https://github.com/JuliaDiffEq/DiffEq

To locally run this tutorial, do the following commands:

```
using DiffEqTutorials
DiffEqTutorials.weave_file("ode_extras","02-feagin.jmd")
```

Computer Information:

```
Julia Version 1.4.2
Commit 44fa15b150* (2020-05-23 18:35 UTC)
Platform Info:
  OS: Linux (x86_64-pc-linux-gnu)
  CPU: Intel(R) Core(TM) i7-9700K CPU @ 3.60GHz
  WORD_SIZE: 64
  LIBM: libopenlibm
  LLVM: libLLVM-8.0.1 (ORCJIT, skylake)
Environment:
  JULIA_DEPOT_PATH = /builds/JuliaGPU/DiffEqTutorials.jl/.julia
  JULIA_CUDA_MEMORY_LIMIT = 536870912
  JULIA_PROJECT = @.
  JULIA_NUM_THREADS = 4
```

Package Information:

```
Status `/builds/JuliaGPU/DiffEqTutorials.jl/tutorials/ode_extras/Project.toml`
[f3b72e0c-5b89-59e1-b016-84e28bfd966d] DiffEqDevTools 2.22.0
[0c46a032-eb83-5123-abaf-570d42b7fbaa] DifferentialEquations 6.14.0
[961ee093-0014-501f-94e3-6117800e7a78] ModelingToolkit 3.11.0
[76087f3c-5699-56af-9a33-bf431cd00edd] NLopt 0.6.0
[2774e3e8-f4cf-5e23-947b-6d7e65073b56] NLsolve 4.4.0
[429524aa-4258-5aef-a3af-852621145aeb] Optim 0.22.0
[1dea7af3-3e70-54e6-95c3-0bf5283fa5ed] OrdinaryDiffEq 5.41.0
[91a5bcdd-55d7-5caf-9e0b-520d859cae80] Plots 1.5.1
[37e2e46d-f89d-539d-b4ee-838fcccc9c8e] LinearAlgebra
[2f01184e-e22b-5df5-ae63-d93ebab69eaf] SparseArrays
```