These notes for the New Keynesian model follow Eric Sims's notes. Section 1 solves the equilibrium conditions of the New Keynesian model, and Section 2 transforms the equilibrium conditions into the desired form for a risk-adjusted linearization.

1 Model

1.1 Household

Households solve the problem

$$\max_{C_t, N_t, B_{t+1}, M_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log \left(\frac{M_t}{P_t} \right) \right)$$

subject to the budget constraint

$$P_t C_t + B_{t+1} + M_t - M_{t-1} \leq W_t N_t + \Pi_t + (1 + i_{t-1}) B_t.$$

In this model, households have demand for money M_t, which is also the numeraire. The price of goods in terms of money is P_t. The stock of nominal bonds a households has is B_t. Note that B_t will be pre-determined at period t while M_t will not be (M_{t-1} is pre-determined). The Lagrangian for the household is

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta} + \theta \log \left(\frac{M_t}{P_t} \right) \right]$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_t (W_t N_t + \Pi_t + (1 + i_{t-1}) B_t - P_t C_t - B_{t+1} - (M_t - M_{t-1})) \right],$$

which implies first-order conditions

$$0 = C_t^{-\sigma} - \lambda_t P_t$$
$$0 = -\psi N_t^{\eta} + \lambda_t W_t$$
$$0 = -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} (1 + i_t)$$
$$0 = \theta \frac{1}{M_t} - \lambda_t + \beta \mathbb{E}_t \lambda_{t+1}.$$

The first two equations can be combined by isolating λ_t. Using $\lambda_t = C_t^{-\sigma}/P_t$, we can obtain the Euler equation for households and an equation relating money balances to consumption.

$$C_t^{-\sigma} \frac{W_t}{P_t} = \psi N_t^{\eta},$$
$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} (1 + i_t),$$
$$\theta \left(\frac{M_t}{P_t} \right)^{-1} = \frac{i_t}{1 + i_t} C_t^{-\sigma}.$$

1.2 Production

Final Producers There is a representative final goods firm which sells consumption goods in a competitive market. It aggregates intermediate goods using the CES technology

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 1$ so that inputs are substitutes. Profit maximization for the final good firm is

$$\max_{Y_t(j)} P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 P_t(j) Y_t(j) \, dj.$$

The FOC for $Y_t(j)$ is

$$0 = P_t \frac{\epsilon}{\epsilon-1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} \frac{\epsilon-1}{\epsilon} Y_t(j)^{-\frac{1}{\epsilon}} - P_t(j)$$

$$0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} - \frac{P_t(j)}{P_t}$$

$$0 = \left(\int_0^1 Y_t(j)^{\frac{\epsilon}{\epsilon-1}} \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon}$$

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

Plugging this quantity into the identity

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) \, dj$$

and simplifying yields the price index

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} \, dj \right)^{\frac{1}{1-\epsilon}}.$$

Intermediate Producers Intermediate goods are producing according to the linear technology

$$Y_t(j) = A_t N_t(j).$$

Intermediate producers minimize cost subject to the constraint of meeting demand and Calvo price rigidities. Formally,

$$\min_{N_t(j)} W_t N_t(j) \qquad \text{s.t.} \qquad A_t N_t(j) \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The Lagrangian is

$$\mathcal{L} = W_t N_t(j) + \varphi_t(j) \left(\left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - A_t N_t(j) \right),$$

so the first-order condition is

$$0 = W_t - \varphi_t(j) A_t \Rightarrow \varphi_t(j) = \frac{W_t}{A_t}.$$

The multiplier φ_t can be interpreted as the nominal marginal cost. Let mc_t be the real marginal cost. Then profits for an intermediate producer is

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j).$$

In addition to the labor choice, firms also have the chance to reset prices in every period with probability $1 - \phi$. This problem can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon} Y_{t+s} \right),$$

where I have imposed that output equals demand. The first-order condition is

$$0 = (1-\epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} (P_{t+s})^{-(1-\epsilon)} Y_{t+s}$$

$$+ \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{u'(C_{t+s})}{u'(C_t)} mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}$$

Divide by $P_t(j)^{-\epsilon}/u'(C_t)$ and re-arrange to obtain

$$P_t(j) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s u'(C_{t+s}) P_{t+s}^{\epsilon-1} Y_{t+s}}.$$

This expression gives the optimal reset price P_t^*, which we can write more compactly as

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{X_{1,t}}{X_{2,t}}$$

where

$$X_{1,t} = u'(C_t) mc_t P_t^{\epsilon} Y_t + \phi\beta \mathbb{E}_t X_{1,t+1}$$
$$X_{2,t} = u'(C_t) P_t^{\epsilon-1} Y_t + \phi\beta \mathbb{E}_t X_{2,t+1}.$$

3

1.3 Equilibrium and Aggregation

To close the model, I assume that the log of technology A_t follows the AR(1)

$$\log A_t = \rho_a \log A_{t_1} + \varepsilon_{a,t},$$

and the growth rate in the log money supply follows the AR(1)

$$\Delta \log M_t = (1 - \rho_m)\pi + \rho_m \Delta \log M_{t-1} + \varepsilon_{m,t},$$

where π is the steady-state rate of inflation. Note that this specification ensures that money balances grow at the same rate as the price level, which ensures real balances are stationary. To re-write the money growth equation in real terms, note that

$$\log(m_t) = \log(M_t) - \log(P_t) \Rightarrow \Delta \log(m_t) = \log(m_t) - \log(m_{t-1}) = \Delta \log(M_t) - \log(1 + \pi_t),$$

hence

$$\Delta \log(m_t) = (1 - \rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1 + \pi_{t-1}) - \log(1 + \pi_t) + \varepsilon_{m,t}.$$

In equilibrium, bond-holding must be zero, hence

$$C_t = w_t N_t + \frac{\Pi_t}{P_t}.$$

Real dividends Π_t satisfy the accounting identity

$$\frac{\Pi_t}{P_t} = \int_0^1 \left(\frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) \right) dj$$
$$= \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj - w_t \int_0^1 N_t(j)\, dj$$

where $w_t = W_t/P_t$. Aggregate labor supply N_t equals aggregate labor demand in equilibrium, and market-clearing for consumption requires

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj = \int_0^1 \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t\, dj = P_t^{\epsilon-1} Y_t \int_0^1 P_t(j)^{1-\epsilon}\, dj = Y_t$$

since $\int_0^1 P_t(j)^{1-\epsilon}\, dj = P_t^{1-\epsilon}$.
The quantity Y_t is aggregate output, so we must have

$$\int_0^1 A_t N_t(j)\, dj = \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t\, dj$$
$$A_t N_t = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} dj = v_t Y_t.$$

4

Thus, aggregate output is

$$Y_t = \frac{A_t N_t}{v_t}.$$

It can be shown that $v_t \geq 1$ by applying Jensen's inequality.

Finally, recall that

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon}\,dj.$$

In each period, a fraction ϕ cannot change their price. Without loss of generality, we may re-order these firms to the top of the interval so that

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon}\,dj.$$

The latter term can be further simplified under the law of large numbers assumption that a positive measure of firms which cannot change their price still comprise a representative sample of all firms, yielding

$$P_t^{1-\epsilon} = (1-\phi)(P_t^*)^{1-\epsilon} + \phi \int_0^1 P_{t-1}(j)^{1-\epsilon}\,dj = (1-\phi)(P_t^*)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Dividing by $P_{t-1}^{1-\epsilon}$ implies

$$(1+\pi_t)^{1-\epsilon} = (1-\phi)(1+\pi_t^*)^{1-\epsilon} + \phi$$

The price dispersion term can similarly be re-written in terms of aggregates by distinguishing which firms get to change prices.

$$
\begin{aligned}
v_t &= \int_0^{1-\phi} \left(\frac{P_t^*}{P_t}\right)^{-\epsilon}\,dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon}\,dj \\
&= \int_0^{1-\phi} \left(\frac{P_t^*}{P_{t-1}}\right)^{-\epsilon}\left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon}\,dj + \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon}\,dj \\
&= (1-\phi)(1+\pi_t^*)^{-\epsilon}(1+\pi_t)^\epsilon + (1+\pi_t)^\epsilon \int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\,dj.
\end{aligned}
$$

By invoking the law of large assumptions applied to any positive measure subset of firms, we must have

$$\int_{1-\phi}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\,dj = \phi \int_0^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon}\,dj = \phi v_{t-1}.$$

Thus, we acquire

$$
\begin{aligned}
v_t &= (1-\phi)(1+\pi_t^*)^{-\epsilon}(1+\pi_t)^\epsilon + \phi(1+\pi_t)^\epsilon v_{t-1} \\
&= (1+\pi_t)^\epsilon\left((1-\phi)(1+\pi_t^*)^{-\epsilon} + \phi v_{t-1}\right).
\end{aligned}
$$

To finish, we need to derive an expression characterizing π_t^*. Define

$$x_{1,t} \equiv \frac{X_{1,t}}{P_t^\epsilon}, \qquad x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon-1}}.$$

It follows that

$$
\begin{aligned}
x_{1,t} &= u'(C_t)mc_t Y_t + \phi\beta\mathbb{E}_t \frac{X_{1,t+1}}{P_t^\epsilon} \\
&= u'(C_t)mc_t Y_t + \phi\beta\mathbb{E}_t \left[\frac{X_{1,t+1}}{P_{t+1}^\epsilon} \frac{P_{t+1}^\epsilon}{P_t^\epsilon} \right] \\
&= u'(C_t)mc_t Y_t + \phi\beta\mathbb{E}_t [x_{1,t+1}(1+\pi_{t+1})^\epsilon] \\
x_{2,t} &= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t \frac{X_{2,t+1}}{P_t^{\epsilon-1}} \\
&= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t \left[\frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}} \frac{P_{t+1}^{\epsilon-1}}{P_t^{\epsilon-1}} \right] \\
&= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t [x_{2,t+1}(1+\pi_{t+1})^{\epsilon-1}].
\end{aligned}
$$

Further,

$$\frac{X_{1,t}}{X_{2,t}} = \frac{x_{1,t}}{x_{2,t}} P_t,$$

hence

$$P_t^* = \frac{\epsilon}{\epsilon-1} \frac{x_{1,t}}{x_{2,t}} P_t$$

$$(1+\pi_t^*) = \frac{\epsilon}{\epsilon-1} \frac{x_{1,t}}{x_{2,t}} (1+\pi_t)$$

All together, the full set of equilibrium conditions are

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{(1 + i_t)}{1 + \pi_{t+1}} \right]$$

$$C_t^{-\sigma} = \psi \frac{N_t^\eta}{w_t}$$

$$m_t = \theta \frac{1 + i_t}{i_t} C_t^\sigma$$

$$mc_t = \frac{w_t}{A_t}$$

$$C_t = Y_t$$

$$Y_t = \frac{A_t N_t}{v_t}$$

$$v_t = (1 + \pi_t)^\epsilon ((1 - \phi)(1 + \pi_t^*)^{-\epsilon} + \phi v_{t-1})$$

$$(1 + \pi_t)^{1-\epsilon} = (1 - \phi)(1 + \pi_t^*)^{1-\epsilon} + \phi$$

$$(1 + \pi_t^*) = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}}(1 + \pi_t)$$

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi \beta \mathbb{E}_t [x_{1,t+1}(1 + \pi_{t+1})^\epsilon]$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta \mathbb{E}_t [x_{2,t+1}(1 + \pi_{t+1})^{\epsilon-1}]$$

$$\log A_t = \rho_a \log(A_{t-1}) + \varepsilon_{a,t}$$

$$\Delta \log(m_t) = (1 - \rho_m)\pi + \rho_m \Delta \log(m_{t-1}) + \rho_m \log(1 + \pi_{t-1}) - \log(1 + \pi_t) + \varepsilon_{m,t}$$

$$\Delta \log m_t = \log m_t - \log m_{t-1},$$

which comprise 14 equations in 14 aggregate variables

$$(C_t, i_t, \pi_t, N_t, w_t, m_t, mc_t, A_t, Y_t, v_t, \pi_t^*, x_{1,t}, x_{2,t}, \Delta \log m_t).$$

Alternatively, the money growth equation can be replaced by the Taylor rule

$$\log(1 + i_t) = (1 - \rho_i) \log(1 + i) + \rho_i \log(1 + i_{t-1}) + (1 - \rho_i)\phi_\pi(\log(1 + \pi_t) - \log(1 + \pi)) + \varepsilon_{i,t},$$

and the third equation relating money demand to consumption could also be ignored. To reduce the number of equations, we utilize this specification. Furthermore, we can also substitute $1 + \pi_t^*$ to remove π_t^* from the aggregate variables.

2   Risk-Adjusted Linearization

We now proceed to converting the equilibrium conditions into a suitable form for a risk-adjusted linearization. The system should conform to the representation

$$0 = \log \mathbb{E}_t \left[\exp \left(\xi(z_t, y_t) + \Gamma_5 z_{t+1} + \Gamma_6 y_{t+1} \right) \right]$$

$$z_{t+1} = \mu(z_t, y_t) + \Lambda(z_t, y_t)(y_{t+1} - \mathbb{E}_t y_{t+1}) + \Sigma(z_t, y_t)\varepsilon_{t+1},$$

where z_t are (predetermined) state variables and y_t are (nondetermined) jump variables. For the remainder of this section, lower case variables are the logs of previously upper case variables, and variables with a tilde are the logs of previously lower case variables (e.g. the real wage w_t)

The first equation becomes

$$1 = \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{1 + i_t}{1 + \pi_{t+1}} \right]$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\log(\beta) - \sigma(c_{t+1} - c_t) + \tilde{i}_t - \tilde{\pi}_{t+1} \right) \right]$$

$$= \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\beta) + \sigma c_t + \tilde{i}_t}_{\xi} - \underbrace{\sigma c_{t+1} - \tilde{\pi}_{t+1}}_{\text{Forward-Looking}} \right) \right],$$

where $c_t = \log(C_t)$, $\tilde{i}_t = \log(1 + i_t)$, and $\tilde{\pi}_t = \log(1 + \pi_{t+1})$.

The second equation becomes

$$1 = \psi \frac{N_t^{\eta}}{C_t^{-\sigma} w_t}$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\underbrace{\log(\psi) + \eta n_t - (-\sigma c_t + \hat{w}_t)}_{\xi} \right) \right],$$

where $n_t = \log(N_t)$ and $\hat{w}_t = \log(w_t)$.

The third equation becomes

$$1 = \frac{w_t}{A_t mc_t}$$

$$0 = \log \mathbb{E}_t \left[\exp \left(\hat{w}_t - a_t - \tilde{mc}_t \right) \right].$$

The fourth and fifth equation become

$$0 = \log \mathbb{E}_t \left[\exp \left(c_t - a_t - n_t + \hat{v}_t \right) \right].$$

The sixth equation becomes

$$0 = \hat{v}_t - \epsilon \tilde{\pi}_t - \log((1 - \phi) \exp(\tilde{\pi}_t^*)^{-\epsilon} + \phi \exp(\hat{v}_{t-1})),$$

where $\hat{v}_{t-1}$ will be treated as an additional state variable, i.e. if $a_t = \hat{v}_t$ is a jump variable and $b_t = \hat{v}_{t-1}$ is a state variable, then

$$b_{t+1} = a_t.$$

The seventh equation becomes

$$0 = (1 - \epsilon) \tilde{\pi}_t - \log((1 - \phi) \exp(\tilde{\pi}_t^*)^{1-\epsilon} + \phi).$$

The eighth equation becomes

$$0 = \tilde{\pi}_t^* - \log\left(\frac{\epsilon}{\epsilon - 1}\right) - \tilde{\pi}_t - (\hat{x}_{1,t} - \hat{x}_{2,t}).$$

By plugging this expression for $\tilde{\pi}_t^*$ into the previous two equations, we can also remove one more variable from the system.

The ninth and tenth equation become

$$1 = \mathbb{E}_t\left[\phi\beta\frac{x_{1,t+1}(1 + \pi_{t+1})}{x_{1,t} - C_t^{-\sigma}mc_t A_t N_t/v_t}\right]$$

$$0 = \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{1,t}) - \exp((1-\sigma)c_t + \tilde{m}c_t))}_{\xi} + \hat{x}_{1,t+1} + \epsilon\tilde{\pi}_{t+1}\right)\right]$$

$$0 = \log\mathbb{E}_t\left[\exp\left(\underbrace{\log(\phi) + \log(\beta) - \log(\exp(\hat{x}_{2,t}) - \exp((1-\sigma)c_t))}_{\xi} + \hat{x}_{2,t+1} + (\epsilon - 1)\tilde{\pi}_{t+1}\right)\right],$$

where the fact that $x_{1,t}$ and $x_{2,t}$ must both be positive implies $x_{1,t} - C_t^{-\sigma}mc_t Y_t$ and $x_{2,t} - C_t^{-\sigma}Y_t$ are both positive, as the expectations on the RHS are also both positive.

For the monetary policy rule, we use $\tilde{i}_{t-1} \equiv \log(1 + i_{t-1})$ and $\varepsilon_{i,t}$ as states and treat i_t as a jump variable, hence

$$\tilde{i}_t = (1 - \rho_i)\tilde{i} + \rho_i\tilde{i}_{t-1} + (1 - \rho_i)\phi_\pi(\tilde{\pi}_t - \tilde{\pi}) + \varepsilon_{i,t}.$$

This formulation allows us to treat the policy rule as an expectational equation.

The above nine equations comprise the expectational equations. The following four equations comprise the states:

$$a_{t+1} = \rho_a a_t + \varepsilon_{a,t+1}$$
$$\hat{v}_{(t-1)+1} = \hat{v}_t$$
$$\tilde{i}_{(t-1)+1} = \tilde{i}_t$$
$$\varepsilon_{i,t+1} = \varepsilon_{i,t+1}$$

To conclude, note that we have three forward difference equations. The first is the Euler equation, which can be expressed as