# Interpolating Normal Splines

Consider the following interpolation problem:

*Problem:* Given points $x_1 < x_2 < \cdots < x_{n_1}$, $s_1 < s_2 < \cdots < s_{n_2}$ and $t_1 < t_2 < \cdots < t_{n_3}$ find a function $f$ such that

$$\begin{aligned} & f(x_i) = u_i\,, \quad i = 1, 2, \dots, n_1\,, \\ & f'(s_j) = v_j\,, \quad j = 1, 2, \dots, n_2\,, \\ & f''(t_k) = w_k\,, \quad k = 1, 2, \dots, n_3\,, \\ & n_1 > 0\,,\ \ n_2 \ge 0\,,\ \ n_3 \ge 0\,. \end{aligned} \tag{1}$$

Note that knots $\{x_i\}$, $\{s_j\}$ and $\{t_k\}$ may coincide.

We assume that function $f$ is an element of Hilbert space $H = H(X)$ over a set $X$ (where $X$ is $R$ or an interval from $R$) and Hilbert space is selected in a such way that it is continuously embedded in space $C^2(X)$ of functions continuous with their second derivatives, therefore functionals $F_i$, $F'_j$, and $F''_k$

$$\begin{aligned} & F_i(\varphi) = \varphi(x_i)\,,\ \ F'_j(\varphi) = \varphi'(s_j)\,,\ \ F''_k(\varphi) = \varphi''(t_k)\,,\ \forall \varphi \in H\,, \\ & x_i,\ s_j,\ t_k \in X\,, \\ & i = 1, 2, \dots, n_1\,,\ \ j = 1, 2, \dots, n_2\,,\ \ k = 1, 2, \dots, n_3\,. \end{aligned}$$

are linear continuous functionals in $H$. It is obvious that all these functionals are linear independent. In accordance with Riesz representation theorem [1] these linear continuous functionals can be represented in the form of inner product of some elements $h_i, h'_j, h''_k \in H$ and $\varphi \in H$, for any $\varphi \in H$:

$$\begin{gathered} f(x_i) = F_i(\varphi) = \langle h_i, \varphi \rangle_H\,, \quad F'_j(\varphi) = \langle h'_j, \varphi \rangle_H\,, \quad F''_k(\varphi) = \langle h''_k, \varphi \rangle_H\,, \quad \forall \varphi \in H\,, \\ i = 1, 2, \dots, n_1\,,\ \ j = 1, 2, \dots, n_2\,,\ \ k = 1, 2, \dots, n_3\,. \end{gathered}$$

Elements $h_i, h'_j$ and $h''_k$ are twice continuously differentiable functions. Thereby the original system of constraints (1) can be written in form:

$$\begin{aligned} & f(x_i) = F_i(f) = \langle h_i, f \rangle_H = u_i\,, \\ & f'(s_j) = F'_j(f) = \langle h'_j, f \rangle_H = v_j\,, \\ & f''(t_k) = F''_k(f) = \langle h''_k, f \rangle_H = w_k\,, \\ & f \in H\,, \\ & i = 1, 2, \dots, n_1\,,\ \ j = 1, 2, \dots, n_2\,,\ \ k = 1, 2, \dots, n_3\,, \end{aligned} \tag{2}$$

here all functions $h_i, h'_j, h''_k \in H$ are linear independent and system of constrains (2) defines a nonempty convex and closed set (as an intersection of hyper-planes in the Hilbert space $H$).

Problem of reconstruction of function $f$ satisfying system of constraints (2) is undetermined. We reformulate it as a problem of finding solution of this system of constraints that has minimal norm:

$$\sigma = \arg\min\{\|f - z\|^2_H : (2), z \in H, \forall f \in H\}\,, \tag{3}$$

where $z \in H$ is a "prototype" function. Solution of this problem exists and it is unique ([6], [16]) as a projection of element $z$ on the nonempty convex closed set in Hilbert space $H$. Element $\sigma$ is an interpolating normal spline.

In accordance with generalized Lagrange method ([13], [16]) solution of the problem (3) can be presented as:

$$\sigma = z + \sum_{i=1}^{n_1} \mu_i h_i + \sum_{j=1}^{n_2} \mu'_j h'_j + \sum_{k=1}^{n_3} \mu''_k h''_k \,, \tag{4}$$

where coefficients $\mu_i$, $\mu'_j$ and $\mu''_k$ are defined by system of linear equations

$$\begin{aligned} \sum_{l=1}^{n_1} g_{il}\mu_l + \sum_{j=1}^{n_2} g'_{ij}\mu'_j + \sum_{k=1}^{n_3} g''_{ik}\mu''_k &= u_i - \langle h_i, z\rangle_H \,, \quad 1 \le i \le n_1 \,, \\ \sum_{i=1}^{n_1} g'_{ij}\mu_i + \sum_{l=1}^{n_2} g''_{jl}\mu'_l + \sum_{k=1}^{n_3} g'''_{jk}\mu''_k &= v_j - \langle h'_j, z\rangle_H \,, \quad 1 \le j \le n_2 \,, \\ \sum_{i=1}^{n_1} g''_{ik}\mu_i + \sum_{j=1}^{n_2} g'''_{jk}\mu'_j + \sum_{l=1}^{n_3} g^{\text{iv}}_{kl}\mu''_l &= w_k - \langle h''_k, z\rangle_H \,, \quad 1 \le k \le n_3 \,, \end{aligned} \tag{5}$$

Matrix of system (5) is the positive definite symmetric Gram matrix of the set of linearly independent elements $\{h_i\}, \{h'_j\}, \{h''_k\}$, and coefficients $g_{il}, g'_{ij}, g''_{ik}, g''_{jl}, g'''_{jk}, g^{\text{iv}}_{kl}$ are defined as follows:

$$\begin{aligned} & g_{il} = \langle h_i, h_l\rangle_H \,, \;\; g'_{ij} = \langle h_i, h'_j\rangle_H \,, \;\; g''_{ik} = \langle h_i, h''_k\rangle_H \\ & g''_{jl} = \langle h'_j, h'_l\rangle_H \,, \;\; g'''_{jk} = \langle h'_j, h''_k\rangle_H \,, \;\; g^{\text{iv}}_{kl} = \langle h''_k, h''_l\rangle_H \,. \end{aligned} \tag{6}$$

Let $H = H(X)$ be a reproducing kernel Hilbert space with reproducing kernel $V(\eta, \xi)$. Recall the definition of the reproducing kernel ([4], [7]). The reproducing kernel is a such function $V(\eta, \xi)$ that

- for every $\xi \in X$, $V(\eta, \xi)$ as function of $\eta$ belongs to $H$
- for every $\xi \in X$ and every function $\varphi \in H$

$$\varphi(\xi) = \langle V(\eta, \xi), \varphi(\eta)\rangle_H \tag{7}$$

Reproducing kernel is a symmetric function:

$$V(\eta, \xi) = V(\xi, \eta) \,,$$

also, in the considered here case it is twice continuously differentiable function by $\xi$ and by $\eta$. Differentiating the identity (7) allows to get the identities for derivatives:

$$\frac{d\varphi(\xi)}{d\xi} = \left\langle \frac{\partial V(\cdot, \xi)}{\partial \xi}, \varphi \right\rangle_H , \quad \frac{d^2\varphi(\xi)}{d\xi^2} = \left\langle \frac{\partial^2 V(\cdot, \xi)}{\partial \xi^2}, \varphi \right\rangle_H . \tag{8}$$

Now it is possible to express functions $h_i, h'_j, h''_k$ via the reproducing kernel $V$. Comparing (2) with (7) and (8) we receive:

$$\begin{aligned} h_i(\eta) &= V(\eta, x_i) \,, \qquad i = 1, 2, \dots, n_1 \\ h'_j(\eta) &= \frac{\partial V(\eta, s_j)}{\partial \xi} \,, \quad j = 1, 2, \dots, n_2 \,, \\ h''_k(\eta) &= \frac{\partial^2 V(\eta, t_k)}{\partial \xi^2} \,, \;\; k = 1, 2, \dots, n_3 \,. \end{aligned} \tag{9}$$

The coefficients (6) of the Gram matrix can be presented as ([7], [8], [10]):

$$\begin{aligned} g_{il} &= \langle h_i, h_l \rangle_H = \langle V(\cdot, x_i), V(\cdot, x_l) \rangle_H = V(x_i, x_l)\,, \\ g'_{ij} &= \langle h_i, h'_j \rangle_H = \left\langle V(\cdot, x_i), \frac{\partial V(\cdot, s_j)}{\partial \xi} \right\rangle_H = \frac{\partial V(x_i, s_j)}{\partial \xi}\,, \\ g''_{ik} &= \langle h_i, h''_k \rangle_H = \left\langle V(\cdot, x_i), \frac{\partial^2 V(\cdot, t_k)}{\partial \xi^2} \right\rangle_H = \frac{\partial^2 V(x_i, t_k)}{\partial \xi^2}\,. \end{aligned} \tag{10}$$

With the help of (7) and (10), we can also calculate $g''_{jl}$ ([8], [10]):

$$\begin{aligned} & g''_{jl} = \langle h'_j, h'_l \rangle_H = \left\langle \frac{\partial V(\cdot, s_j)}{\partial \xi}, \frac{\partial V(\cdot, s_l)}{\partial \xi} \right\rangle_H = \\ & \frac{d}{d\xi} \left\langle V(\cdot, \xi), \frac{\partial V(\cdot, s_l)}{\partial \xi} \right\rangle_H \Bigg|_{\xi = s_j} = \\ & \frac{d}{d\xi} \left( \frac{\partial V(\xi, s_l)}{\partial \xi} \right) \Bigg|_{\xi = s_j} = \frac{\partial^2 V(s_j, s_l)}{\partial \eta \partial \xi}\,. \end{aligned} \tag{11}$$

Further

$$\begin{aligned} g'''_{jk} &= \langle h'_j, h''_k \rangle_H = \frac{\partial^3 V(s_j, t_k)}{\partial \eta \partial \xi^2}\,, \\ g^{\mathrm{iv}}_{kl} &= \langle h''_k, h''_l \rangle_H = \frac{\partial^4 V(t_k, t_l)}{\partial \eta^2 \partial \xi^2}\,, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \langle h_i, z \rangle_H &= \langle V(\cdot, x_i), z \rangle_H = z(x_i)\,, \\ \langle h'_j, z \rangle_H &= z'(s_j)\,, \\ \langle h''_k, z \rangle_H &= z''(t_k)\,. \end{aligned} \tag{13}$$

Normal splines will be constructed in Sobolev space $W^3_2[a, b]$ and in Bessel potential space $H^3_\varepsilon(R)$ (See [1, 2, 4, 20] for details). Elements of these spaces can be treated as twice continuously differentiable functions.

Reproducing kernel for Sobolev spaces $W^l_2[0, 1]$ (here $l$ — any positive integer) was constructed in work [8]. Thus, reproducing kernel for Sobolev space $W^3_2[0, 1]$ with norm

$$\|f\|_{W^3_2} = \left( \sum_{i=0}^{2} (f^{(i)}(0))^2 + \int_0^1 (f^{(3)}(s))^2 ds \right)^{1/2}\,,$$

can be presented as

$$V(\eta, \xi) = \begin{cases} \sum_{i=0}^{2} \frac{\xi^i}{!i} \left( \frac{\eta^i}{!i} + (-1)^i \frac{\eta^{5-i}}{(5-i)!} \right), & 0 \le \eta \le \xi \le 1 \\ \sum_{i=0}^{2} \frac{\eta^i}{!i} \left( \frac{\xi^i}{!i} + (-1)^i \frac{\xi^{5-i}}{(5-i)!} \right), & 0 \le \xi \le \eta \le 1 \end{cases}$$

or

$$V(\eta, \xi) = \begin{cases} 1 + \eta\xi + \frac{(\eta^5 - 5\eta^4\xi + 10\eta^3\xi^2 + 30\eta^2\xi^2)}{120}, & 0 \le \eta \le \xi \le 1 \\ 1 + \eta\xi + \frac{(\xi^5 - 5\xi^4\eta + 10\xi^3\eta^2 + 30\xi^2\eta^2)}{120}, & 0 \le \xi \le \eta \le 1 \end{cases} \tag{14}$$

Correspondingly

$$
\begin{aligned}
& \frac{\partial V(\eta,\xi)}{\partial \xi} = \frac{\eta(4\eta\xi(\eta+3) - \eta^3)}{24} + \eta \,, \\
& \frac{\partial^2 V(\eta,\xi)}{\partial \eta \partial \xi} = -\frac{\eta^3}{6} + \frac{\eta\xi(\eta+2)}{2} + 1 \,, \\
& \frac{\partial^2 V(\eta,\xi)}{\partial \xi^2} = \frac{\eta^2(\eta+3)}{6} \,, \\
& \frac{\partial^3 V(\eta,\xi)}{\partial \eta \partial \xi^2} = \frac{\eta^2}{2} + \eta \,, \\
& \frac{\partial^4 V(\eta,\xi)}{\partial \eta^2 \partial \xi^2} = \eta + 1 \,, \\
& 0 \le \eta \le \xi \le 1 \,.
\end{aligned}
\tag{15}
$$

In addition, the following formulae are required for computing the normal spline derivatives

$$
\begin{aligned}
& \frac{\partial V(\eta,\xi)}{\partial \eta} = \frac{\eta^4 - 4\xi(\eta^3 - 6) + 6\eta\xi^2(\eta+2)}{24} \,, \\
& \frac{\partial^2 V(\eta,\xi)}{\partial \eta^2} = \frac{\eta^3 - 3\eta^2\xi + 3\xi^2(\eta+1)}{6} \,, \\
& \frac{\partial^3 V(\eta,\xi)}{\partial \eta^2 \partial \xi} = -\frac{\eta^2}{2} + \eta\xi + \xi \,, \\
& 0 \le \eta \le \xi \le 1 \,.
\end{aligned}
\tag{16}
$$

Thereby we can construct a normal interpolating spline in interval $[0,1]$. Solving the interpolating problem in an arbitrary interval can be done by mapping the latter to $[0,1]$ through affine change of variable. Define constants $a$ and $b$ as

$$
a = \min(x_1, s_1, t_1), \qquad b = \max(x_{n_1}, s_{n_2}, t_{n_3}) \,,
$$

and introduce values $\bar{x}_i, \bar{s}_j, \bar{t}_k$:

$$
\begin{aligned}
& \bar{x}_i = \frac{x_i - a}{b - a}, \quad \bar{s}_j = \frac{s_j - a}{b - a}, \quad \bar{t}_k = \frac{t_k - a}{b - a} \,, \\
& i = 1, \dots, n_1, \quad j = 1, \dots, n_2, \quad k = 1, \dots, n_3 \,.
\end{aligned}
$$

Then original Problem (1) is transformed to

*Problem:* Given points $0 \le \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_{n_1} \le 1$, $0 \le \bar{s}_1 < \bar{s}_2 < \cdots < \bar{s}_{n_2} \le 1$ and $0 \le \bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_{n_3} \le 1$ find a function $\bar{f}$ such that

$$
\begin{aligned}
& \bar{f}(\bar{x}_i) = u_i \,, \quad i = 1, 2, \dots, n_1 \,, \\
& \bar{f}'(\bar{s}_j) = v_j(b-a) \,, \quad j = 1, 2, \dots, n_2 \,, \\
& \bar{f}''(\bar{t}_k) = w_k(b-a)^2 \,, \quad k = 1, 2, \dots, n_3 \,.
\end{aligned}
\tag{17}
$$

Assuming $\bar{\sigma}(\bar{\eta})$ is a normal spline constructed for the Problem (17), the normal spline $\sigma(\eta)$ can be received as

$$
\sigma(\eta) = \bar{\sigma}\left(\frac{\eta - a}{b - a}\right), \qquad a \le \eta \le b \,.
$$

Reproducing kernel for Bessel potential space was presented in [5] and its simplified variant in [14], [3], [18], [19]. Here normal splines will be constructed in Bessel potential space $H^3_\varepsilon(R)$ defined as

$$H^3_\varepsilon(R) = \left\{ f | f \in S', (\varepsilon^2 + |s|^2)^{3/2}\mathcal{F}[f] \in L_2(R) \right\} , \varepsilon > 0,$$

where $S'(R)$ is space of L. Schwartz tempered distributions and $\mathcal{F}[f]$ is a Fourier transform of the $f$ ([5], [3], [17]). The parameter $\varepsilon$ may be treated as a "scaling parameter". It allows to control approximation properties of the normal spline which usually are getting better with smaller values of $\varepsilon$, also it may be used to reduce the illconditioness of the related computational problem (in traditional theory $\varepsilon = 1$) ([14], [20]). Space $H^3_\varepsilon(R)$ is a Hilbert space with norm

$$\|f\|_{H^3_\varepsilon} = \|(\varepsilon^2 + |s|^2)^{3/2}\mathcal{F}[\varphi]\|_{L_2} \ ,$$

it is continuously embedded in the Hölder space $C^3_b(R)$ that consists of all functions having bounded continuous derivatives up to order 2 ([3]). The reproducing kernel of this space is defined up to a constant as follows ([14], [20])

$$V(\eta, \xi, \varepsilon) = \frac{1}{\varepsilon^2} \exp(-\varepsilon|\xi - \eta|)(3 + 3\varepsilon|\xi - \eta| + \varepsilon^2|\xi - \eta|^2)\,. \tag{18}$$

Correspondingly

$$\begin{aligned}
\frac{\partial V(\eta,\xi,\varepsilon)}{\partial \xi} &= -\exp(-\varepsilon|\xi-\eta|)(\xi-\eta)(\varepsilon|\xi-\eta|+1)\,,\\
\frac{\partial^2 V(\eta,\xi,\varepsilon)}{\partial \eta \partial \xi} &= -\exp(-\varepsilon|\xi-\eta|)(\varepsilon|\xi-\eta|(\varepsilon|\xi-\eta|-1)-1)\,,\\
\frac{\partial^2 V(\eta,\xi,\varepsilon)}{\partial \xi^2} &= \exp(-\varepsilon|\xi-\eta|)(\varepsilon|\xi-\eta|(\varepsilon|\xi-\eta|-1)-1)\,,\\
\frac{\partial^3 V(\eta,\xi,\varepsilon)}{\partial \eta \partial \xi^2} &= \varepsilon^2 \exp(-\varepsilon|\xi-\eta|)(\xi-\eta)(\varepsilon|\xi-\eta|-3)\,,\\
\frac{\partial^2 V(\eta,\xi,\varepsilon)}{\partial \eta^2 \partial \xi^2} &= \varepsilon^2 \exp(-\varepsilon|\xi-\eta|)(\varepsilon|\xi-\eta|(\varepsilon|\xi-\eta|-5)+3)\,.
\end{aligned} \tag{19}$$

In addition, the following formulae are required for computing the normal spline derivatives

$$\begin{aligned}
\frac{\partial V(\eta,\xi,\varepsilon)}{\partial \eta} &= \exp(-\varepsilon|\xi-\eta|)(\xi-\eta)\varepsilon|\xi-\eta|+1)\,,\\
\frac{\partial^2 V(\eta,\xi,\varepsilon)}{\partial \eta^2} &= \exp(-\varepsilon|\xi-\eta|)(\varepsilon|\xi-\eta|(\varepsilon|\xi-\eta|-1)-1)\,,\\
\frac{\partial^3 V(\eta,\xi,\varepsilon)}{\partial \eta^2 \partial \xi} &= -\varepsilon^2 \exp(-\varepsilon|\xi-\eta|)(\xi-\eta)(\varepsilon|\xi-\eta|-3)\,.
\end{aligned} \tag{20}$$

In a case when there is no information of second derivative values of function $f$ the Problem (1) is reducing to the following one:

*Problem:* Given points $x_1 < x_2 < \dots < x_{n_1}$, $s_1 < s_2 < \dots < s_{n_2}$ find a function $f$ such that

$$\begin{aligned}
& f(x_i) = u_i\,, \quad i = 1, 2, \dots, n_1\,,\\
& f'(s_j) = v_j\,, \quad j = 1, 2, \dots, n_2\,,\\
& n_1 \ge 0\,,\ \ n_2 \ge 0\,.
\end{aligned} \tag{21}$$

We assume here that $f$ is a continuously differentiable function. It can be treated as an element of Sobolev space $W^2_2[a,b]$ or Bessel potential space $H^2_\varepsilon(R)$. First space is continuously embedded in space $C^1[a,b]$ of functions continuous with their first derivative and the second one is continuously embedded in Hölder space $C^1_b(R)$ of continuously differentiable functions with bounded derivative.

Reproducing kernel of Sobolev spaces $W^2_2[0,1]$ with norm

$$\|f\|_{W^2_2} = \left( \sum_{i=0}^{1} (f^{(i)}(0))^2 + \int_0^1 (f^{(2)}(s))^2 ds \right)^{1/2} ,$$

was received in [11]:

$$V(\eta,\xi) = \begin{cases} 1 + (\eta + \eta^2/2)\xi - \eta^3/6 , & 0 \le \eta \le \xi \le 1 \\ 1 + (\xi + \xi^2/2)\eta - \xi^3/6 , & 0 \le \xi \le \eta \le 1 \end{cases} \tag{22}$$

Correspondingly

$$\begin{aligned} &\frac{\partial V(\eta,\xi)}{\partial \xi} = \eta^2/2 + \eta , \\ &\frac{\partial^2 V(\eta,\xi)}{\partial \eta \partial \xi} = \eta + 1 , \\ &0 \le \eta \le \xi \le 1 . \end{aligned} \tag{23}$$

In addition, the following formula is required for computing the normal spline derivative

$$\begin{aligned} &\frac{\partial V(\eta,\xi)}{\partial \eta} = -\eta^2/2 + \eta\xi + \xi , \\ &0 \le \eta \le \xi \le 1 . \end{aligned} \tag{24}$$

Reproducing kernel for Bessel potential space $H^2_\varepsilon(R)$ with norm

$$\|f\|_{H^2_\varepsilon} = \|(\varepsilon^2 + |s|^2)\mathcal{F}[\varphi]\|_{L_2} .$$

is defined by ([14], [20]):

$$V(\eta,\xi,\varepsilon) = \frac{1}{\varepsilon^2} \exp(-\varepsilon|\xi - \eta|)(1 + \varepsilon|\xi - \eta|) . \tag{25}$$

Correspondingly

$$\begin{aligned} &\frac{\partial V(\eta,\xi,\varepsilon)}{\partial \xi} = -\exp(-\varepsilon|\xi - \eta|)(\xi - \eta) , \\ &\frac{\partial^2 V(\eta,\xi,\varepsilon)}{\partial \eta \partial \xi} = \exp(-\varepsilon|\xi - \eta|)(1 - \varepsilon|\xi - \eta|) . \end{aligned} \tag{26}$$

In addition, the following formula is required for computing the normal spline derivative

$$\frac{\partial V(\eta,\xi,\varepsilon)}{\partial \eta} = \exp(-\varepsilon|\xi - \eta|)(\xi - \eta) . \tag{27}$$

In a case when there is no information of function $f$ derivatives the Problem (1) is reducing to the simplest interpolation problem:

*Problem:* Given points $x_1 < x_2 < \cdots < x_{n_1}$ find a function $f$ such that

$$\begin{aligned} & f(x_i) = u_i \,, \quad i = 1, 2, \dots, n_1 \,, \\ & n_1 > 0 \,. \end{aligned} \tag{28}$$

We assume that $f$ is a continuous function. It can be treated as an element of Sobolev space $W^1_2[a,b]$ or Bessel potential space $H^1_\varepsilon(R)$. First space is continuously embedded in space $C[a,b]$ of continuous functions and the second one is continuously embedded in Hölder space $C^{1/2}_b(R)$ of continuous and bounded functions.

Reproducing kernel of Sobolev spaces $W^2_1[0,1]$ with norm

$$\|f\|_{W^1_2} = \left( (f^2(0) + \int_0^1 (f^{(1)}(s))^2 ds \right)^{1/2} \,,$$

was received in [11]:

$$V(\eta,\xi) = \begin{cases} 1+\eta \,, & 0 \le \eta \le \xi \le 1 \\ 1+\xi \,, & 0 \le \xi \le \eta \le 1 \end{cases} \tag{29}$$

Reproducing kernel for Bessel potential space $H^1_\varepsilon(R)$ with norm

$$\|f\|_{H^1_\varepsilon} = \|(\varepsilon^2 + |s|^2)^{1/2} \mathcal{F}[\varphi]\|_{L_2} \,.$$

is defined by ([14], [20]):

$$V(\eta,\xi,\varepsilon) = \exp(-\varepsilon|\xi-\eta|) \,. \tag{30}$$

The normal splines method for one-dimensional function interpolation and linear ordinary differential and integral equations was proposed in [11] and developed in [8], [9], [10]. Multidimensional normal splines method was developed for two-dimensional problem of low-range computerized tomography in [15] and applied for solving a mathematical economics problem in [12]. Further results were reported on seminars and conferences.

**References**

[1] R. Adams, J. Fournier, Sobolev Spaces. Pure and Applied Mathematics. (2nd ed.). Boston, MA: Academic Press, 2003.

[2] D. Adams, L. Hedberg, Function spaces and potential theory. Berlin, Springer, 1996.

[3] M.S. Agranovich, Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains, Springer, Switzerland, 2015.

[4] N. Aronszajn, Theory of reproducing kernels, Tranzactions of the AMS.– 950 – Vol.68.

[5] N. Aronszajn, K.T. Smith, Theory of bessel potentials I, Ann.Inst.Fourier, 11, 1961.

[6] A. Balakrishnan, Applied Functional Analysis, New York, Springer-Verlag, 1976.

[7] A. Bezhaev, V. Vasilenko, Variational Theory of Splines, Springer US, 2001.

[8] V. Gorbunov, V. Petrishchev, Improvement of the normal spline collocation method for linear differential equations, Comput. Math. Math. Phys., 43:8 (2003), 1099–1108

[9] V. Gorbunov, V. Sviridov, The method of normal splines for linear DAEs on the number semi-axis. Applied Numerical Mathematics, 59(3-4), 2009, 656–670.

[10] V. Gorbunov, Extremum Problems of Measurements Data Processing, Ilim, 1990. (in Russian)

[11] V. Gorbunov, The method of normal spline collocation, USSR Computational Mathematics and Mathematical Physics,Volume 29, Issue 1, 1989, Pages 145-154.

[12] V. Gorbunov, I. Kohanovsky, K. Makedonsky, Normal splines in reconstruction of multi-dimensional dependencies. Papers of WSEAS International Conference on Applied Mathematics, Numerical Analysis Symposium, Corfu, 2004.

[13] A. Ioffe, V. Tikhomirov, Theory of extremal problems, North-Holland, Amsterdam, 1979.

[14] I. Kohanovsky, Multidimensional Normal Splines and Problem of Physical Field Approximation, International Conference on Fourier Analysis and its Applications, Kuwait, 1998.

[15] I. Kohanovsky, Normal Splines in Computing Tomography, Avtometriya, 1995, N 2.

[16] P.-J. Laurent, Approximation et optimization, Paris, 1972.

[17] H. Triebel, Interpolation. Function Spaces. Differential Operators. North-Holland, Amsterdam, 1978.

[18] R. Schaback, Kernel-based Meshless Methods, Lecture Notes, Goettingen, 2011.

[19] H. Wendland, Scattered Data Approximation. Cambridge University Press, 2005.

[20] Reproducing Kernel of Bessel Potential space.