Orthogonal Polynomials on and in Algebraic Curves and Surfaces

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Joint work with
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OVERVIEW

- Computing with curves?
- Multivariate orthogonal polynomials
- Nonclassical domains:
  - Half-disks, trapeziums
  - Circles, arcs, spheres and polar caps (?)
  - Quadratic surfaces of revolution

- What is the “right” way to represent curves and surfaces?
    - “Right” implies spectrally accurate
- What is the “right” way to do function approximation on and inside curves and surfaces?
    - “Right” implies spectrally accurate, a la Chebyshev expansion

Use algebraic curves/surfaces!

- Approximate general curves/surfaces by algebraic curves/surfaces
  - That is zero sets of polynomials
- Use restrictions of polynomials to algebraic curves for function approximation
  - Polynomials modulo the vanishing ideal
  - Orthogonalizing gives multivariate orthogonal polynomials with nice structure

Representing Curves, 3 Options

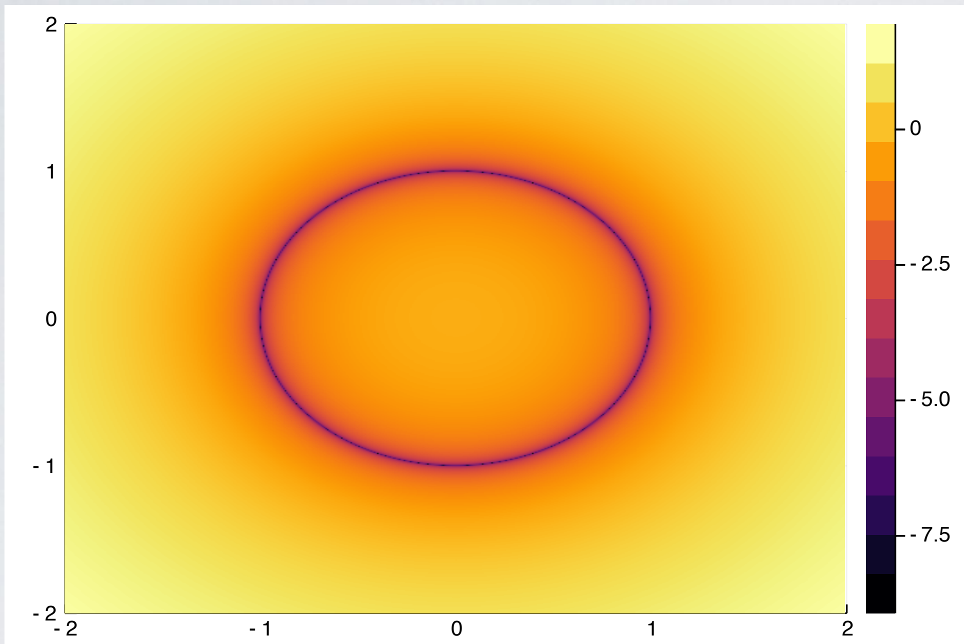
- Grid points + Interpolation
- Parameterisation
- Level set method

Representing Curves, 3 Options

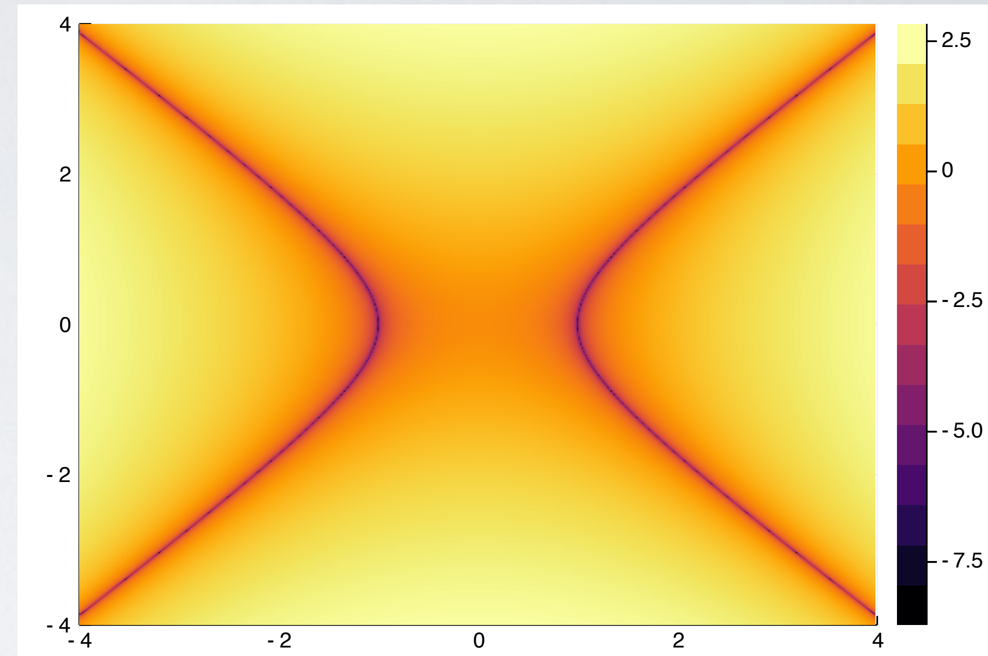
- Grid points + Interpolation
- Parameterisation
- Level set method

using polynomials

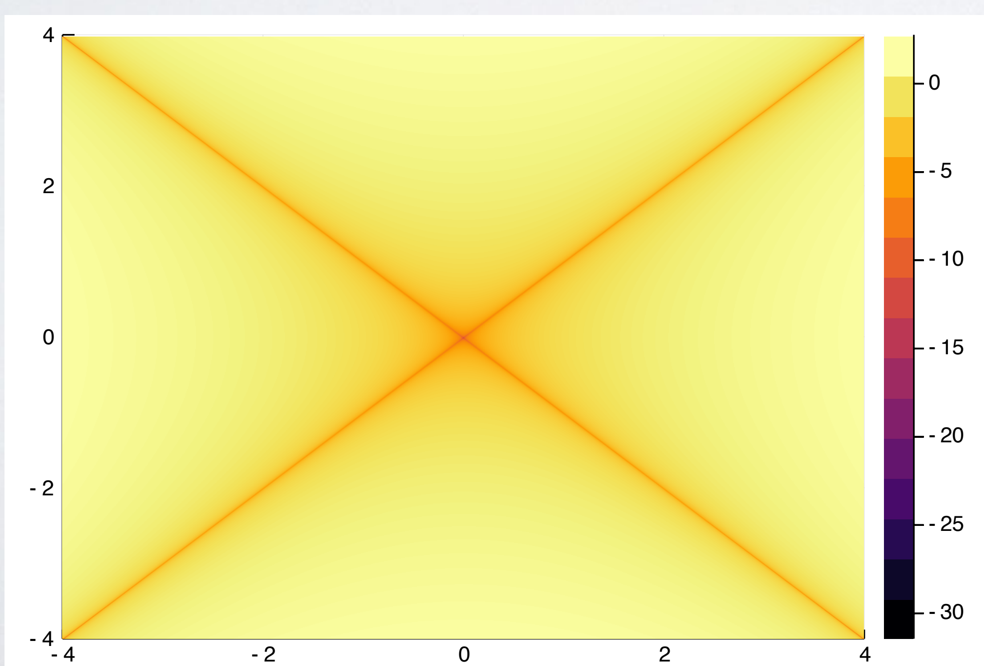
$$x^2 + y^2 - 1$$



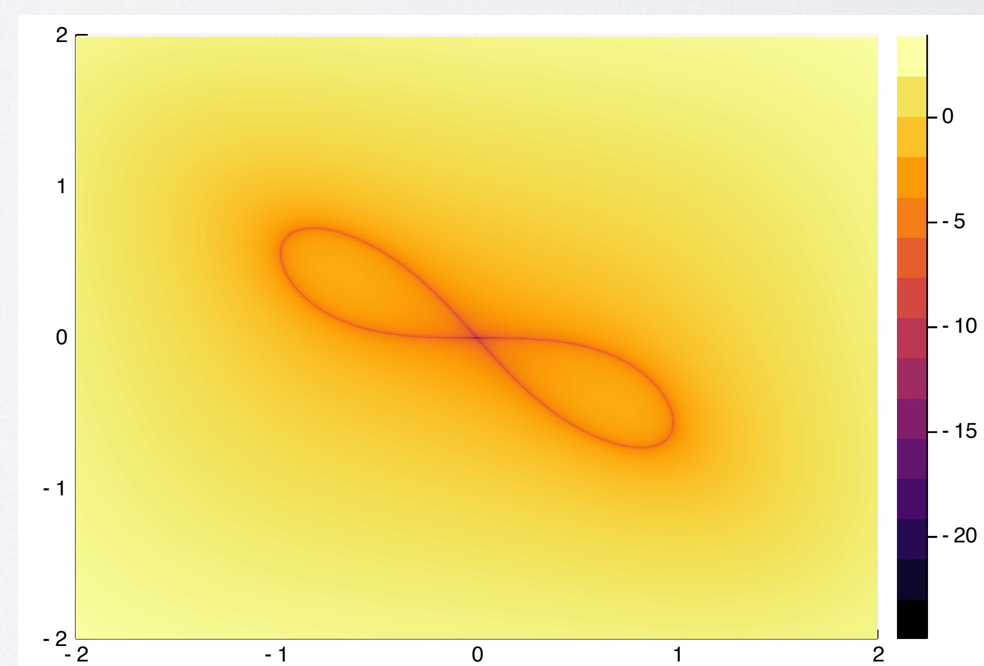
$$x^2 - y^2 - 1$$



$$x^2 - y^2$$



$$x^4 + 3xy + 2y^2 + y^4$$



Approximation of Curves and Surfaces by Algebraic Curves and Surfaces

PA Smith - Annals of Mathematics, 1926 - JSTOR

00 (1) z~~~~~~ An (x, y) n= 1 which will converge uniformly in a region R containing J to a continuous function which is 0 on J, and which, in R1 is> 0 at points exterior to J and< 0 at interior points. The series (1) moreover is to give rise (by equating successive sums to zero) to a sequence of non-singular algebraic ovals converging to J in a manner explicitely described in Theorem 1. Analogous results will be obtained for (n-1) dimensional manifolds in n-space (for example, simple closed surfaces in 3-space) but only for a restricted class of …

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0 Citations!

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[HTML] On the approximation of convex bodies by convex **algebraic level surfaces**

A Kroó - Journal of Approximation Theory, 2010 - Elsevier

In this note we consider the problem of the approximation of convex bodies in R d by level surfaces of convex algebraic polynomials. Hammer (1963)[1] verified that any convex body in R d can be approximated by a level surface of a convex algebraic polynomial. In Kroó …

Cited by 3 Related articles All 6 versions Web of Science: 3

[PDF] **Approximate implicitization**

T Dokken - Mathematical methods for curves and surfaces, 2001 - researchgate.net

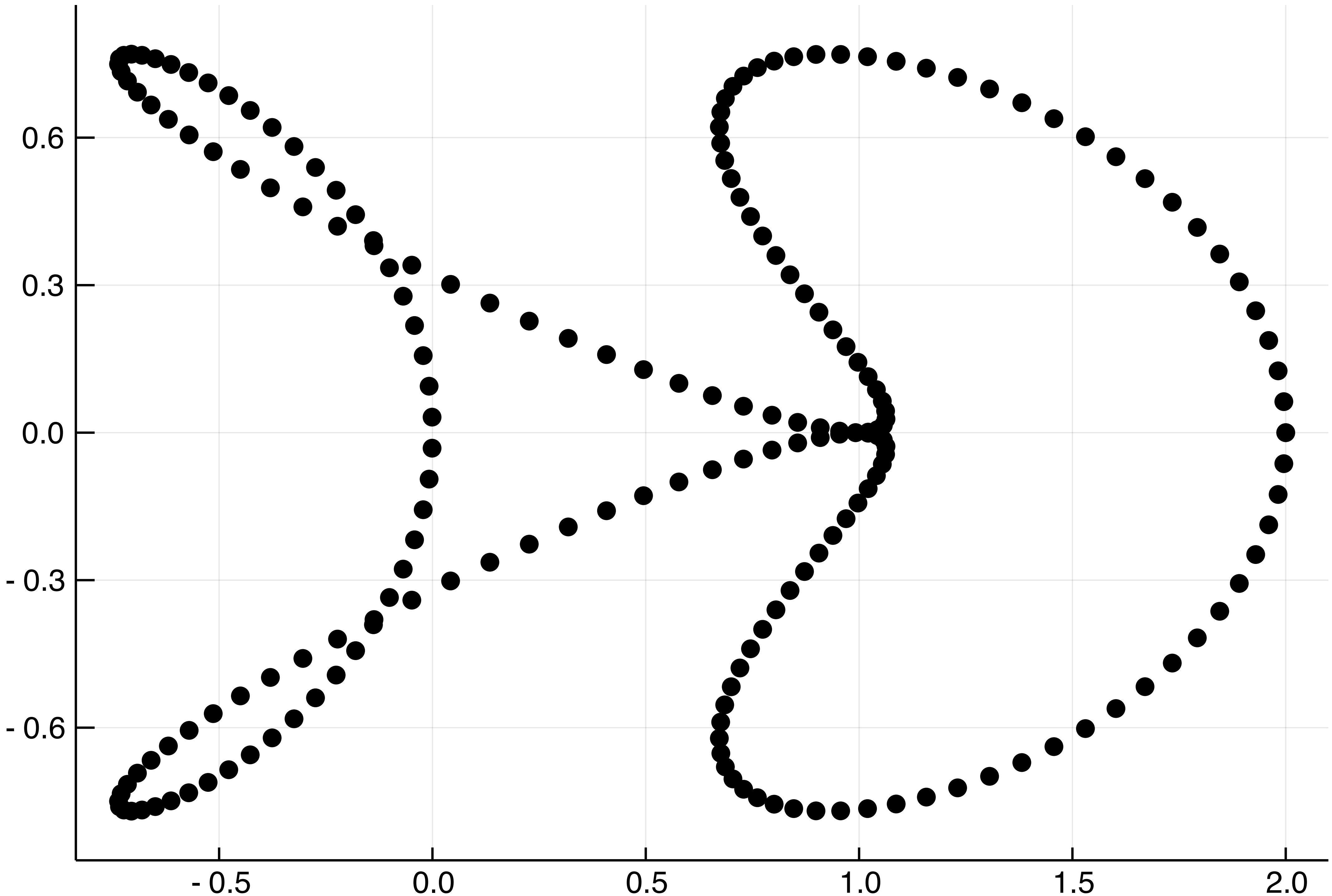
… Page 10. Implicit Surfaces and Algebraic Distance The intention is to find a polynomial q describing an implicit surface that **approximates** . / in the tetrahedral Bernstein basis of degree m q 0 " C +++"m b+Bm + 0 . The task is to find the unknown values b+ for +++ " m that satisfy …

Cited by 88 Related articles All 4 versions

Approximate Implicitization

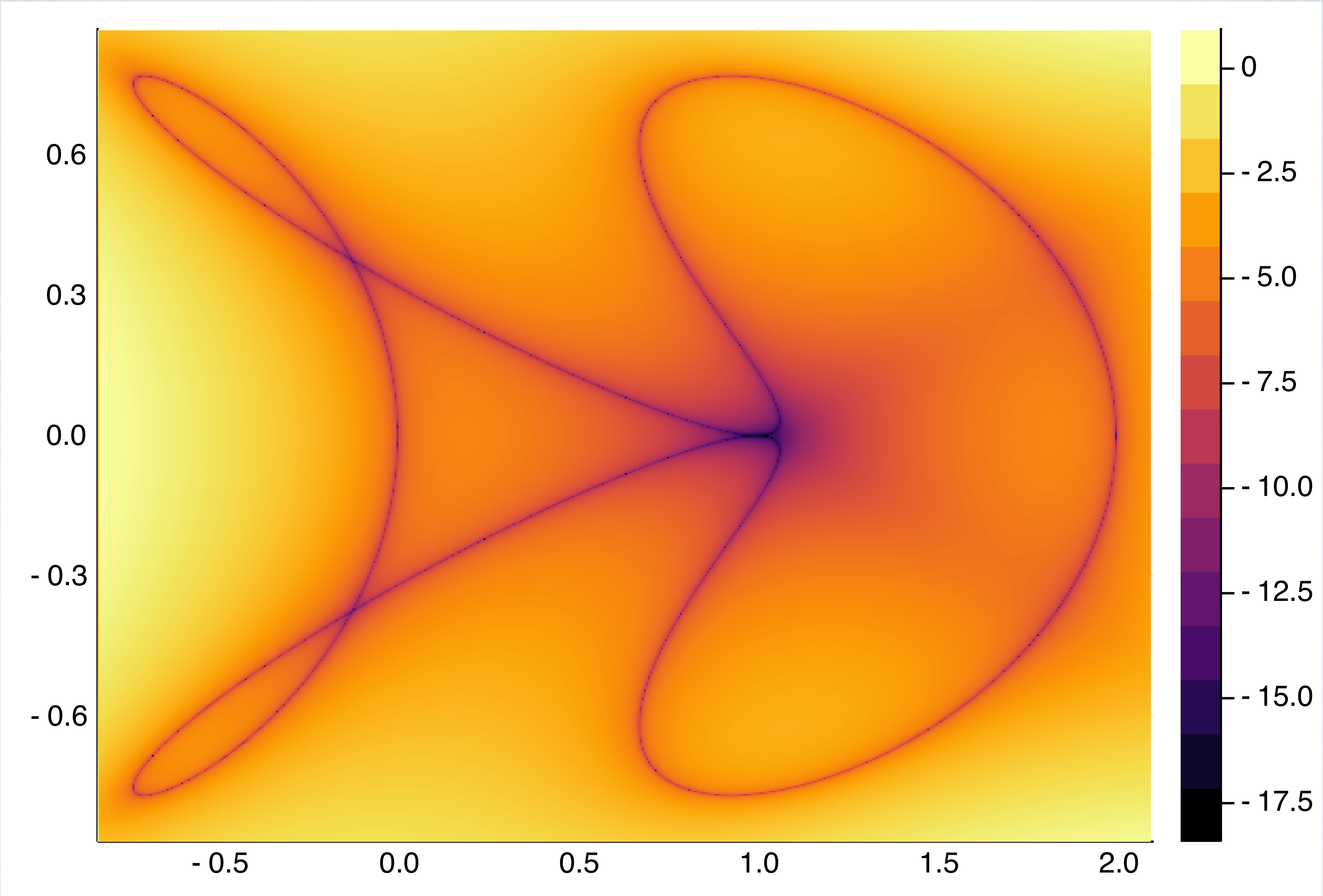
- Usually, curves are given by points $(x_1, y_1), \ldots, (x_m, y_m)$
- There exists an easy way to numerically calculate a polynomial $p(x, y)$ whose zero set approximates the desired curve:
- Embed the curve in a square, and represent $p(x, y)$ as a degree n tensored Chebyshev expansion
- Construct the evaluation matrix at the points
- The null space of this matrix gives the coefficients of $p(x, y)$
- Adaptively increase n until there is a nullspace

Approximate Implicitisation



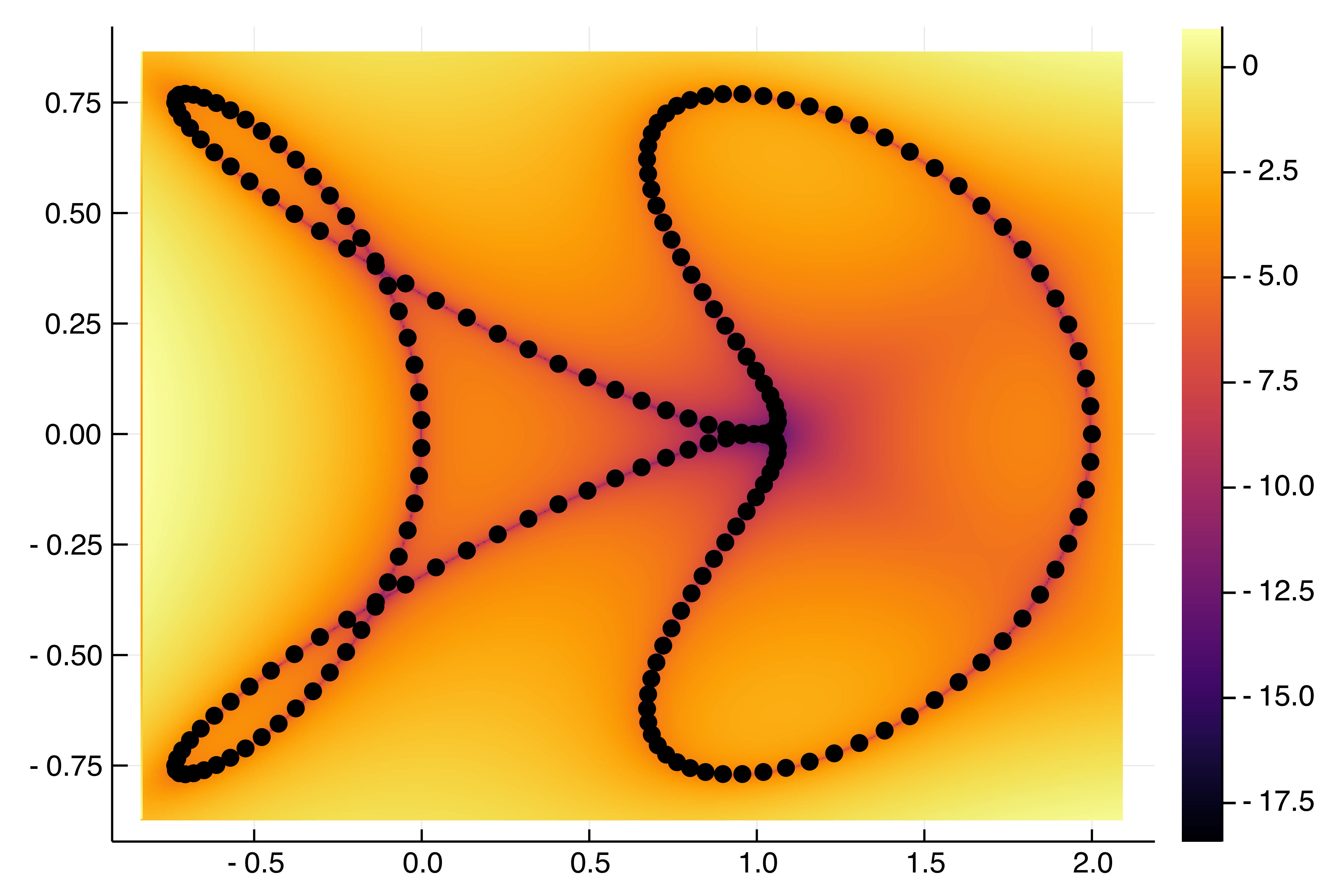
Points given on a curve

Approximate Implicitisation

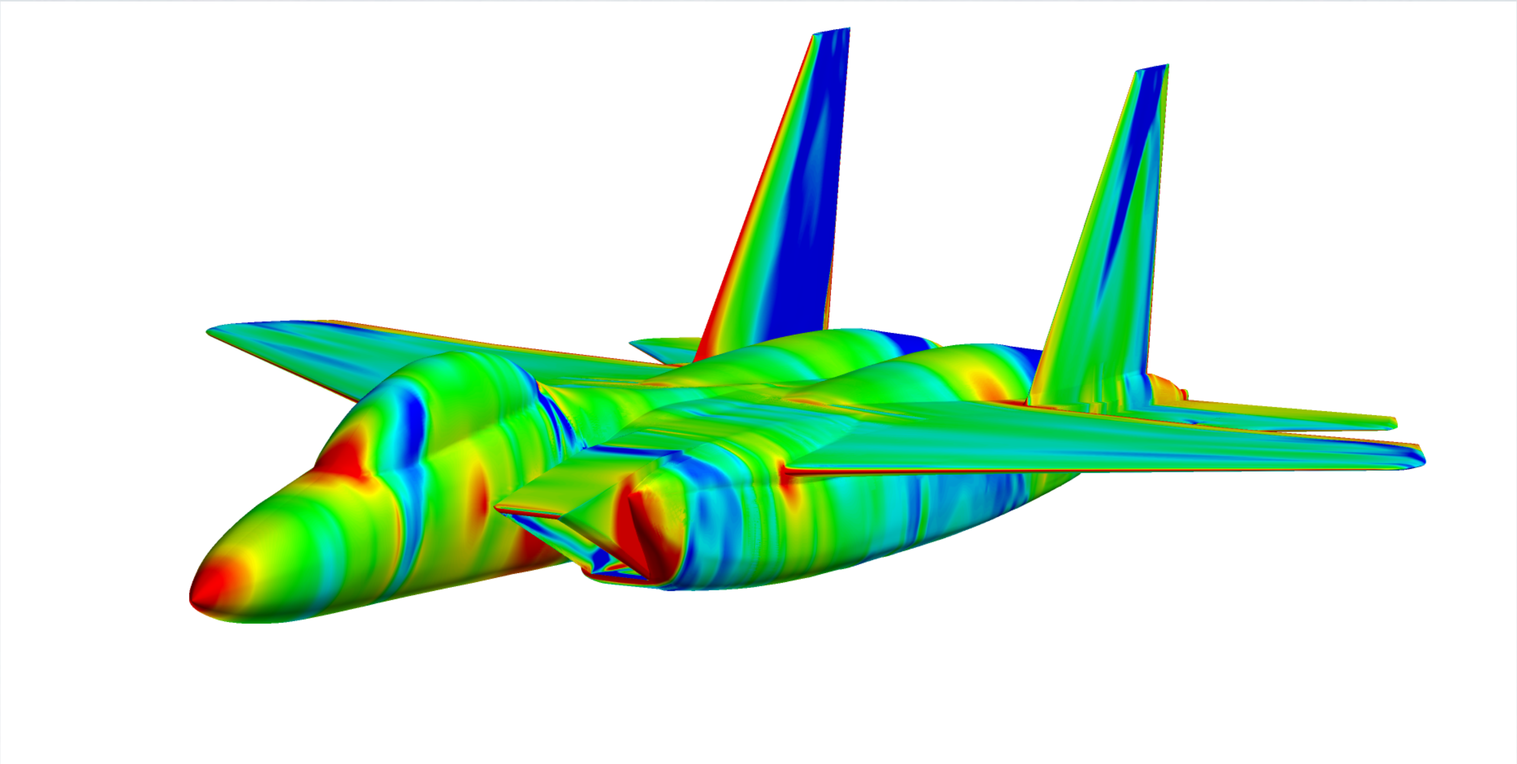


The calculated polynomial in tensor Chebyshev

APPROXIMATE IMPLICITISATION



The zero set of the polynomial match the given points: its in the kernel of the Vandermonde matrix

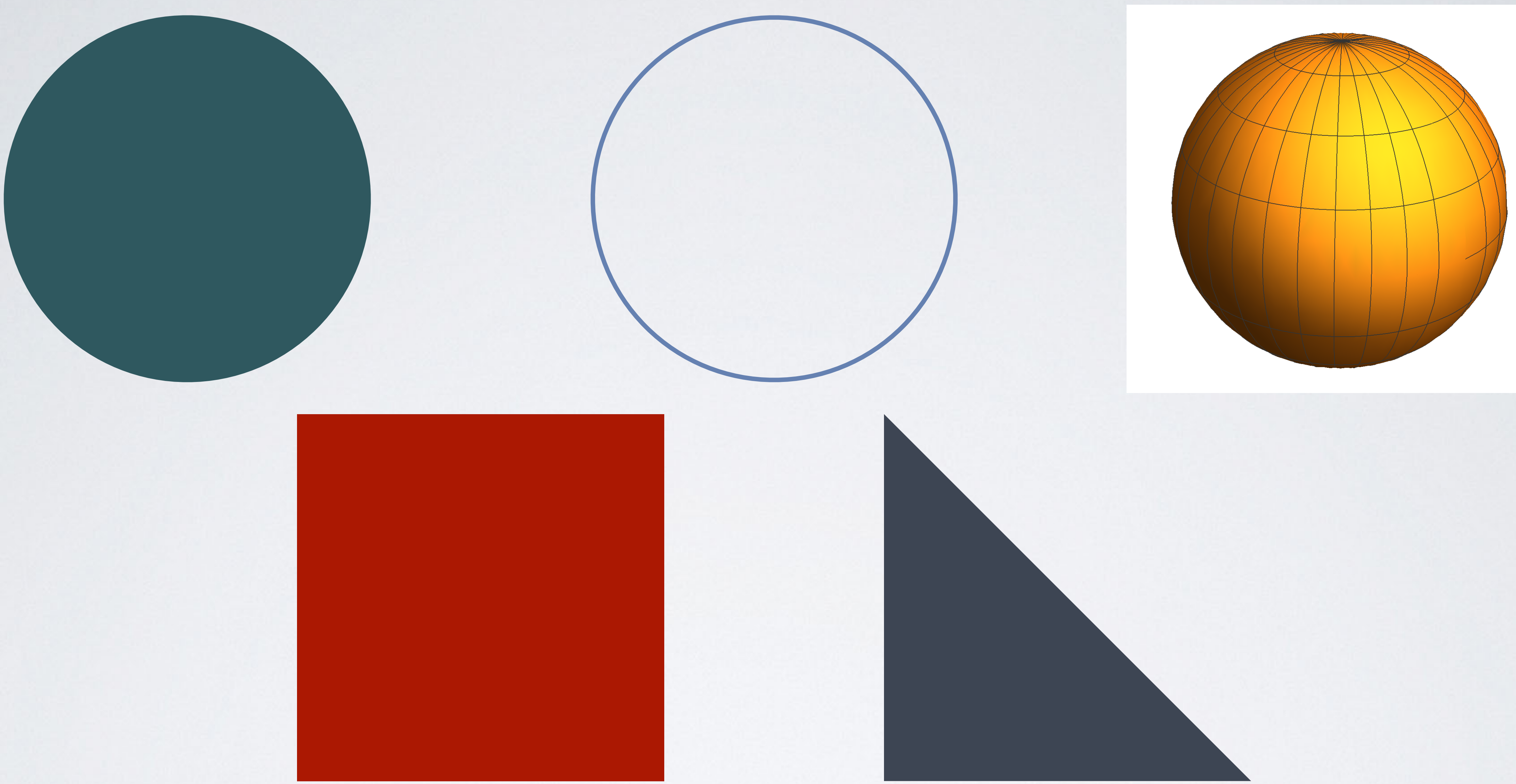


- Long term goal: computational methods on complicated domains as algebraic curves
  - Solve PDEs on surfaces, inside surfaces, etc. using orthogonal polynomials a la ultraspherical spectral method
  - Work with geometries defined via nonuniform rational B-splines (NURBS)
- Short term goal: do something (anything!) with OPs on and inside simple algebraic curves and surfaces

Multivariate orthogonal polynomials

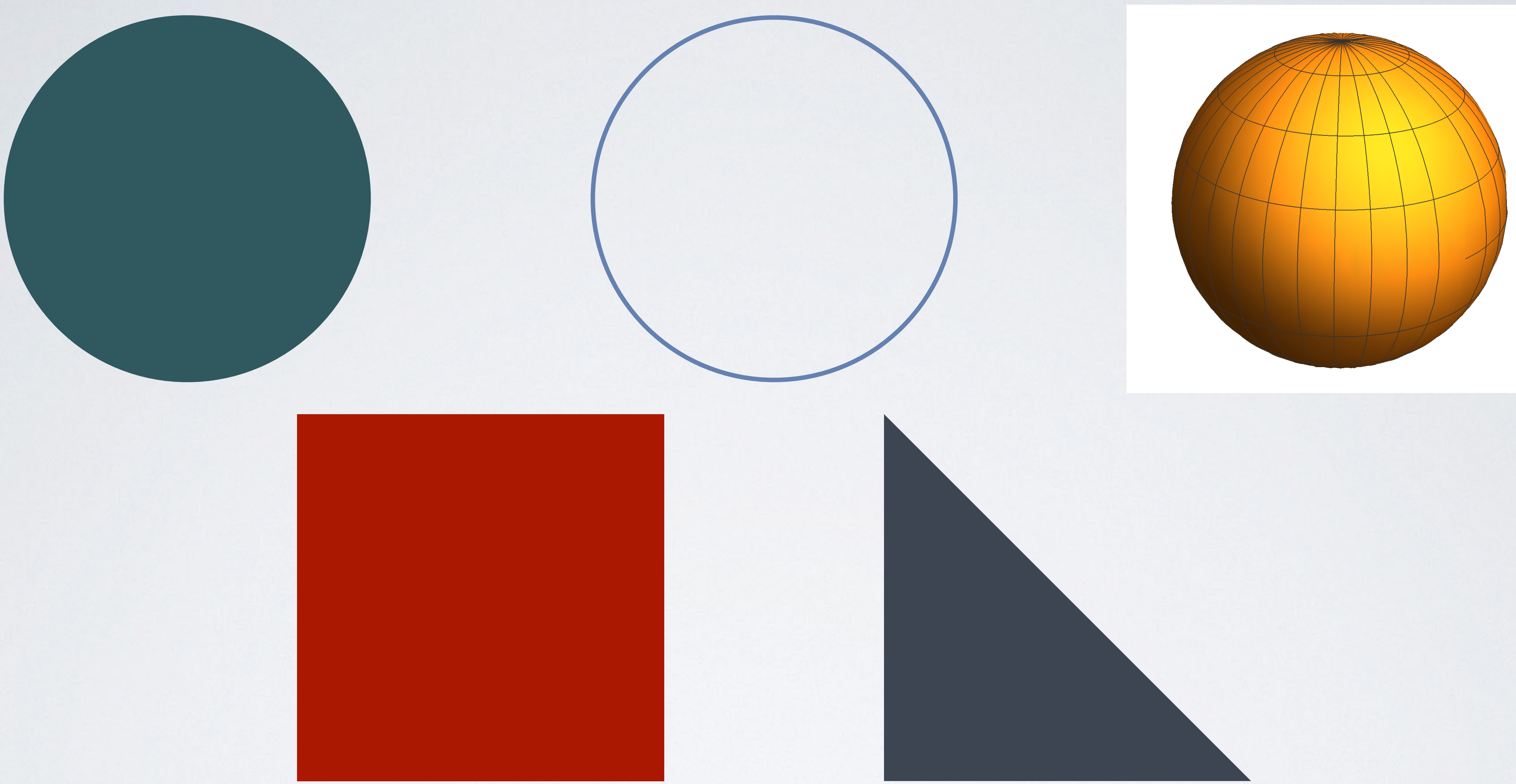
[Dunkl & Xu 2014]

CLASSICAL



Classical multivariate orthogonal polynomials
allow function approximation and
solving PDEs on
balls, circles, squares, triangles, spheres

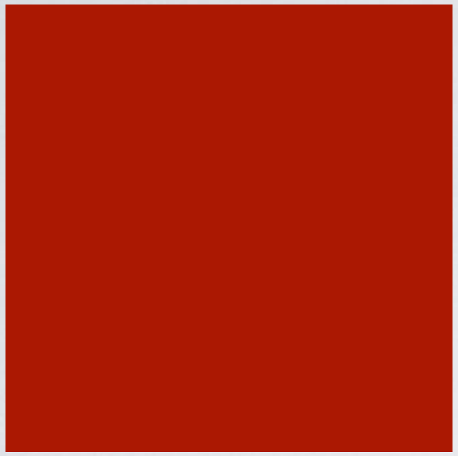
CLASSICAL



Fast transforms in software thanks to Slevinsky

2D OPs are (kinda) like 1D OPs

- Non-uniqueness
- Three-term recurrence
- Jacobi operators
- Evaluation
- Clenshaw and multiplication operators

Domain	Weight	OPs
	$\sqrt{1-x^2}\sqrt{1-y^2}$	$T_{n-k}(x)T_k(y)$
	$x^a y^b (1-x-y)^c$	$P_{n-k}^{(2k+b+c+1,a)}(x)(1-x)^k \times P_k^{(c,b)}\left(\frac{2y}{1-x}-1\right)$
	1	$U_n\left(x\cos\frac{k\pi}{n+1}+y\sin\frac{k\pi}{n+1}\right)$

- Consider an inner product on $\Omega \subset \mathbb{R}^2$ of the form

$$\langle f, g \rangle = \int_\Omega f(x, y) g(x, y) w(x, y) \, \mathrm{d}V$$

- Consider orthonormal polynomials $P_{nk}(x, y)$, $k = 0, \ldots, n$ with respect to this inner product

NON-UNIQENESS

- In 1D, OPs are only uniquely defined up to sign
  - If $p_n(x)$ are orthogonal so are $\pm p_n(x)$ for any choice of signs
- In 2D, OPs are only defined up to orthogonal transformations
  - For any $Q_n \in O(n+1)$, $Q_n \mathbb{P}_n$ are also orthonormal polynomials:

$$\left\langle Q_n \mathbb{P}_n, (Q_m \mathbb{P}_m)^\top \right\rangle = Q_n \left\langle \mathbb{P}_n, \mathbb{P}_m^\top \right\rangle Q_m^\top = \begin{cases} I_n & n = m \\ 0_{n \times m} & n \neq m \end{cases}$$

Three-term Recurrences

- In 1D, OPs satisfy three-term recurrences:

$$xp_n(x) = c_{n-1}p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x)$$

  – This follows since for $m < n-1$,

$$\langle xp_n, p_m\rangle = \langle p_n, xp_m\rangle = 0$$

- In 2D, OPs satisfy two block three-term recurrences: for $A_n^x, A_n^y \in \mathbb{R}^{(n+1)\times(n+1)}$, $B_n^x, B_n^y \in \mathbb{R}^{(n+1)\times(n+2)}$, $C_n^x, C_n^y \in \mathbb{R}^{(n+2)\times(n+1)}$

$$\begin{aligned} x\mathbb{P}_n(x,y) &= C_{n-1}^x \mathbb{P}_{n-1}(x,y) + A_n^x \mathbb{P}_n(x,y) + B_n^x \mathbb{P}_{n+1}(x,y) \\ y\mathbb{P}_n(x,y) &= C_{n-1}^y \mathbb{P}_{n-1}(x,y) + A_n^y \mathbb{P}_n(x,y) + B_n^y \mathbb{P}_{n+1}(x,y) \end{aligned}$$

  – This follows since for $m < n-1$,

$$\begin{aligned} \left\langle x\mathbb{P}_n, \mathbb{P}_m^\top\right\rangle &= \left\langle \mathbb{P}_n, x\mathbb{P}_m^\top\right\rangle = 0 \\ \left\langle y\mathbb{P}_n, \mathbb{P}_m^\top\right\rangle &= \left\langle \mathbb{P}_n, y\mathbb{P}_m^\top\right\rangle = 0 \end{aligned}$$

JACOBI OPERATORS

- 1D OPs have symmetric tridiagonal Jacobi operators:

$$J\begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = x\begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix}, \qquad J = \begin{pmatrix} a_0 & b_0 & & \\ c_0 & a_1 & b_1 & \\ & c_1 & a_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

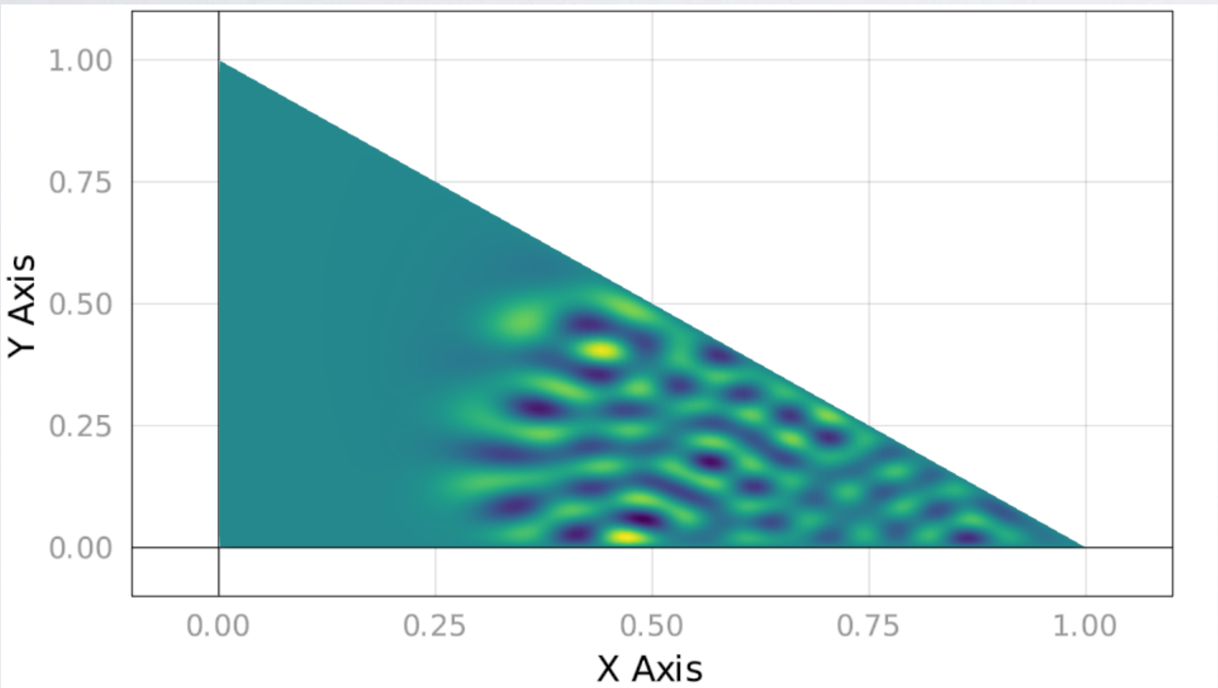
- 2D OPs have a pair of commuting operators J_x and J_y satisfying

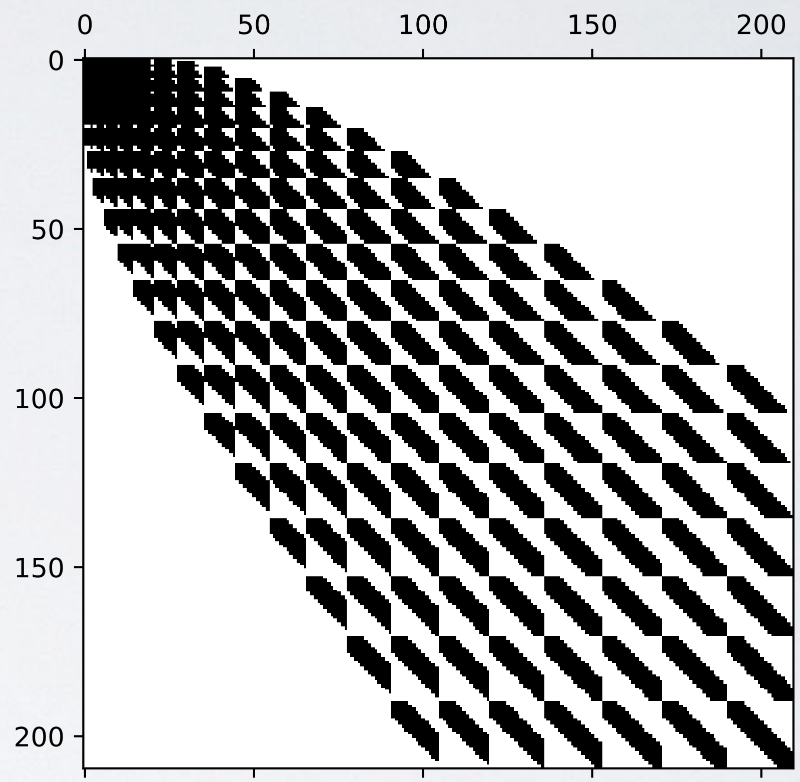
$$J_x\begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix} = x\begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix} \quad \text{and} \quad J_y\begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix} = y\begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix}.$$

- Here, J_x and J_y are block tridiagonal symmetric operators:

$$J_x = \begin{pmatrix} A_0^x & B_0^x & & \\ C_0^x & A_1^x & B_1^x & \\ & C_1^x & A_2^x & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad J_y = \begin{pmatrix} A_0^y & B_0^y & & \\ C_0^y & A_1^y & B_1^y & \\ & C_1^y & A_2^y & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

OPs lead to sparse discretisations of PDEs



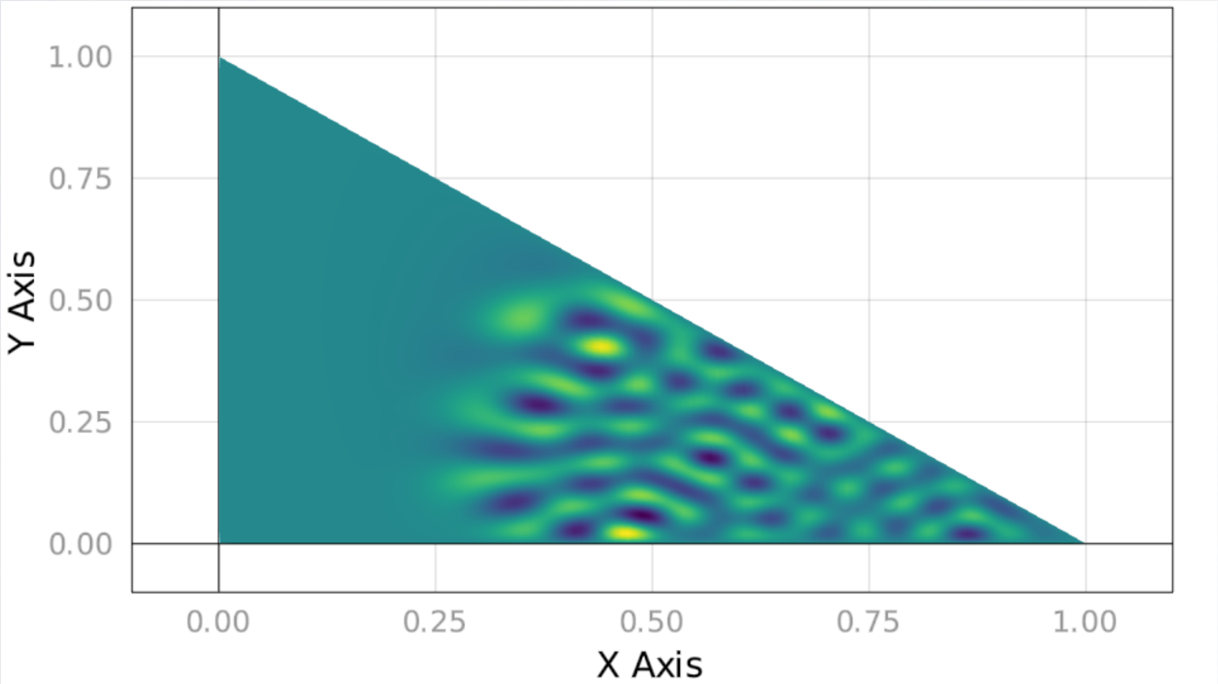


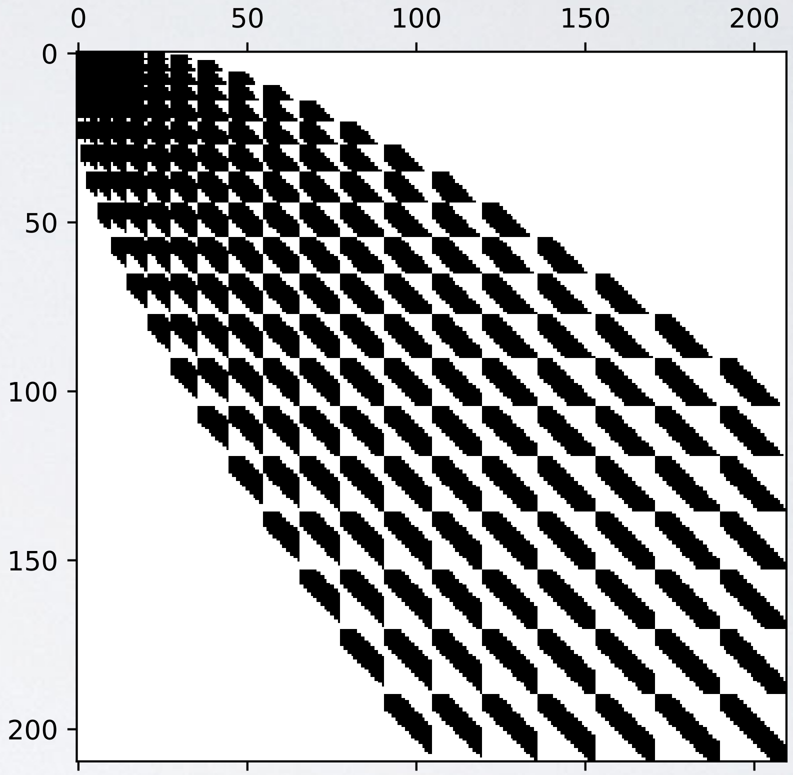
$$\Delta u + v(x, y)u = f$$

Degree 2-polynomial

[Beuchler & Schoeberl 2006]
[Li & Shen 2010]
[SO, Townsend & Huybrechs 2019]

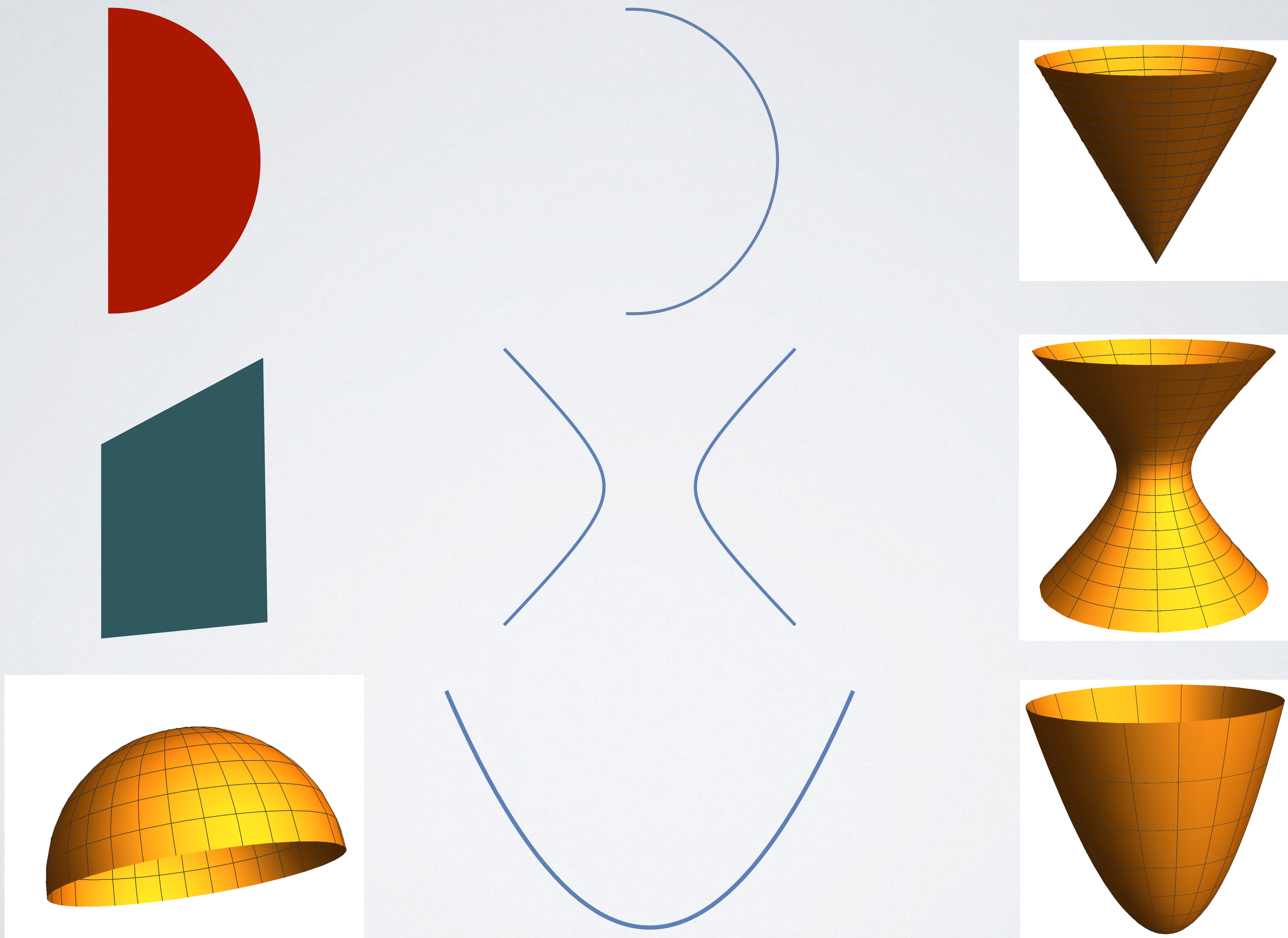
OPs lead to sparse discretisations of PDEs





Sparsity not specific to a triangle: guaranteed because boundary is algebraic curve!

Non-classical?



Evaluation

- 1D OPs can be constructed via forward recurrence build from the Jacobi operator

$$L_x \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} e_0^\top \\ J - xI \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ a_0 - x & b_0 & & \\ c_0 & a_1 - x & b_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

- However, 2D OPs have too much information:

$$\tilde{L}_{x,y} \begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ A_0^x - xI & B_0^x & & \\ A_0^y - yI & B_0^y & & \\ C_0^x & A_1^x - xI & B_1^x & \\ C_0^y & A_1^y - yI & B_1^y & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

- We rectify this by finding a pseudo-inverse

$$\left(D_n^x \mid D_n^y \right) \begin{pmatrix} B_n^x \\ B_n^y \end{pmatrix} = I_{n+1}$$

- Define

$$R = \begin{pmatrix} 1 & & & \\ & \left(D_0^x \mid D_0^y \right) & & \\ & & \left(D_1^x \mid D_1^y \right) & \\ & & & \ddots \end{pmatrix}$$

- We obtain a lower triangular system

$$L_{x,y} \begin{pmatrix} \mathbb{P}_0(x,y) \\ \mathbb{P}_1(x,y) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

for

$$\begin{aligned} L_{x,y} &= R\tilde{L}_{x,y} \\ &= \begin{pmatrix} 1 & & & \\ D_0^x A_0^x - x D_0^x + D_0^y A_0^y - y D_0^y & I & & \\ D_1^x C_0^x + D_1^y C_0^y & D_1^x A_1^x - x D_1^x + D_1^y A_1^y - y D_1^y & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \end{aligned}$$

CLENSHAW'S ALGORITHM

- Evaluating an expansion is thus a back-substitution:

$$f(x,y) = \left(\mathbb{P}_0(x,y)^\top, \mathbb{P}_1(x,y)^\top \ldots\right)^\top \begin{pmatrix} \boldsymbol{f}_0 \\ \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \end{pmatrix} = e_0^\top L_{x,y}^{-\top} \begin{pmatrix} \boldsymbol{f}_0 \\ \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \end{pmatrix}$$

- Constructing multiplication operators follows also by:

$$f(J_x, J_y)$$

  - A recurrence of block-banded operator multiplications, that gives a block-banded operator

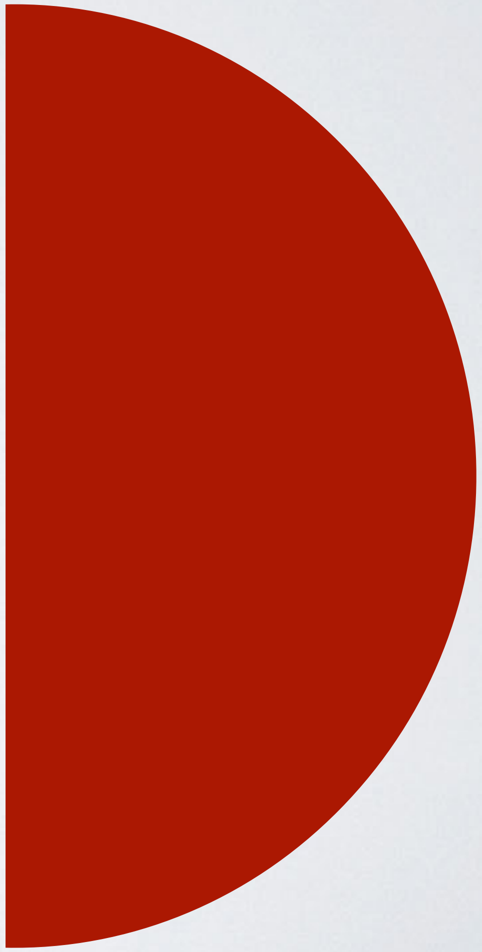
HALF-DISK

- We can construct OPs ortho. w.r.t. $x^a(1-x^2-y^2)^b$ on a half-disk:

$$H_{n,k}^{(a,b)}(x,y) := R_{n-k}^{(a,b+k+\frac{1}{2})}(x)\,(1-x^2)^{k/2}\,P_k^{(b,b)}\left(\frac{y}{\sqrt{1-x^2}}\right)$$

  where $R_k^{(a,b)}(x)$ are ortho. w.r.t. $x^a(1-x^2)^{b/2}$ on $0 < x < 1$

- Example of domain whose boundary is an algebraic curve
- Generalizes to trapeziums and disk-slices



HALF-DISK

Surprisingly hard to construct (help!)

- We can construct OPs ortho. w.r.t. $x^a(1-x^2-y^2)^b$ on a half-disk:

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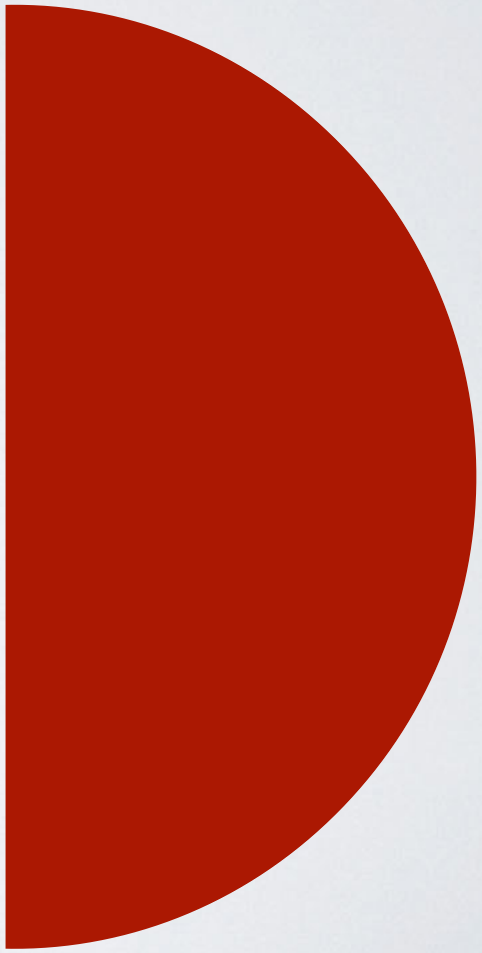
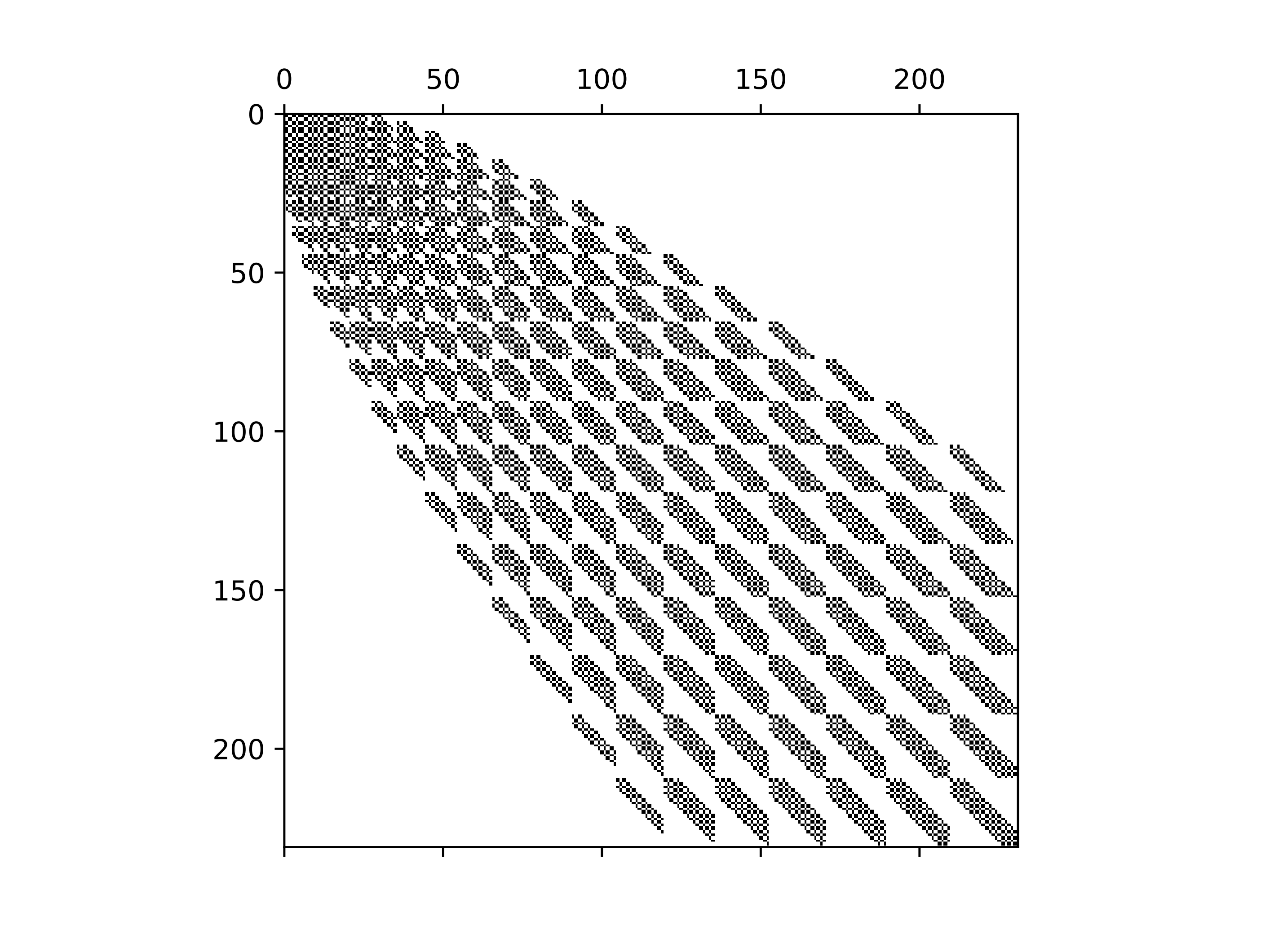
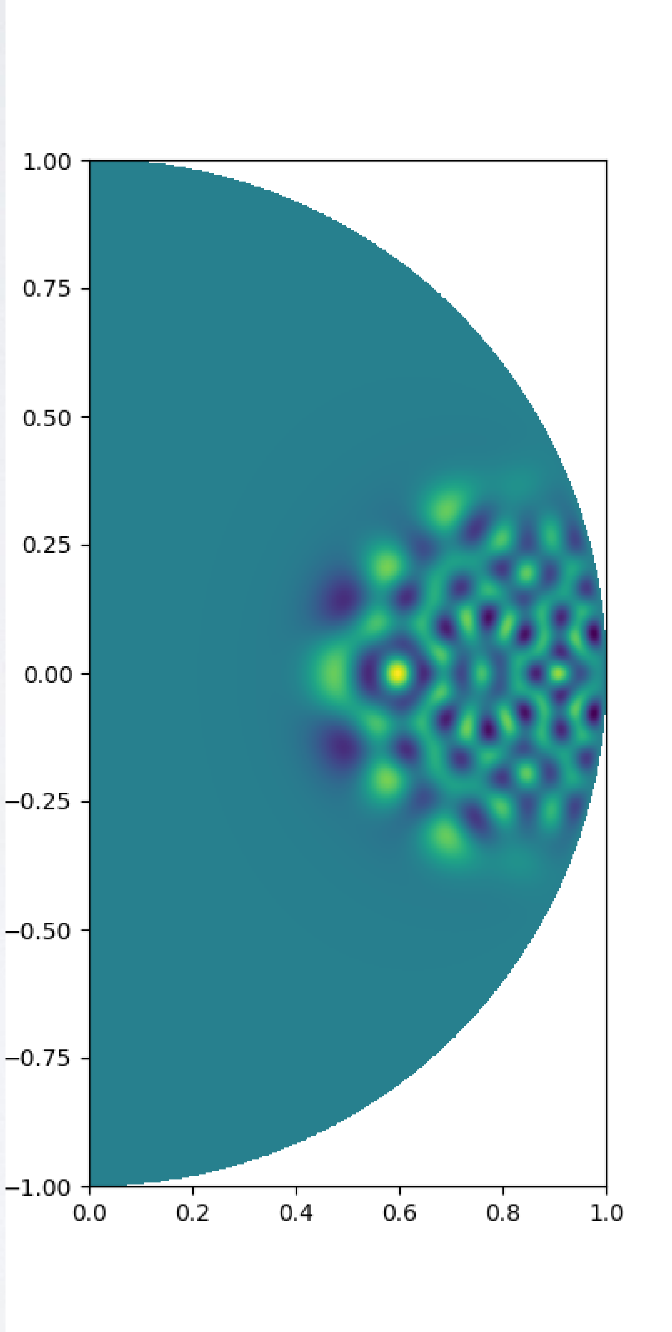


OPs lead to sparse discretisations of PDEs





$$\Delta u + v(x,y)u = f$$

[Snowball & SO 2019]

Circles And Arcs

- What about inner products on curves?

$$\langle f, g \rangle = \int_\Gamma f(x,y) g(x,y) w(x,y) \, \mathrm{d}s$$

- When Γ is an algebraic curve it is the root of a polynomial
- The dimension of the degree n polynomials collapses
- But the structure of OPs is still there!
  - Three-term recurrences, Jacobi operators, etc.
- For special curves and weights, we can express in terms of 1D OPs

CIRCLES

Polynomials on a Circle

$$1$$

$$x$$
$$y$$

$$x^2$$
$$xy$$
$$y^2$$

$$x^3$$
$$x^2y$$
$$xy^2$$
$$y^3$$

$$\vdots$$

Polynomials on a Circle

$$
\begin{array}{ll}
1 & \\
\hline
x & \\
y & \\
\hline
x^2 & \\
xy & \\
y^2 & = 1 - x^2 \\
\hline
x^3 & \\
x^2y & \\
xy^2 & = x - x^3 \\
y^3 & = y - yx^2 \\
\hline
\vdots &
\end{array}
$$

Polynomials on a Circle

$$1$$

$$x$$
$$y$$

$$x^2$$
$$xy$$
$$\cancel{y^2 = 1 - x^2}$$

$$x^3$$
$$x^2y$$
$$\cancel{xy^2 = x - x^3}$$
$$\cancel{y^3 = y - yx^2}$$

$$\vdots$$

Polynomials on a Circle

$$1$$

$$x$$
$$y$$

$$x^2$$
$$xy$$
$$\cancel{y^2 = 1 - x^2}$$

$$x^3$$
$$x^2y$$
$$\cancel{xy^2 = x - x^3}$$
$$\cancel{y^3 = y - yx^2}$$

$$\vdots$$

Dimension 2

OPs on the Circle (uniform weight)

1

---

x

y

$2x^2 - 1$

$2xy$

$4x^3 - 3x$

$4x^2y - y$

$8x^4 - 8x^2 + 1$

$8x^3y - 4xy$

$\vdots$

OPs on the Circle (uniform weight)

$$
\begin{array}{c}
1 \\
\hline
x \\
y \\
\hline
2x^2 - 1 \\
2xy \\
\hline
4x^3 - 3x \\
4x^2 y - y \\
\hline
8x^4 - 8x^2 + 1 \\
8x^3 y - 4xy \\
\hline
\vdots
\end{array}
\quad \equiv \quad
\begin{array}{c}
T_0(x) \\
\hline
T_1(x) \\
yU_0(x) \\
\hline
T_2(x) \\
yU_1(x) \\
\hline
T_3(x) \\
yU_2(x) \\
\hline
T_4(x) \\
yU_3(x) \\
\hline
\vdots
\end{array}
$$

OPs on the Circle (uniform weight)

$$
\begin{array}{c} 1 \\ \hline x \\ y \\ \hline 2x^2-1 \\ 2xy \\ \hline 4x^3-3x \\ 4x^2y-y \\ \hline 8x^4-8x^2+1 \\ 8x^3y-4xy \\ \hline \vdots \end{array}
\equiv
\begin{array}{c} T_0(x) \\ \hline T_1(x) \\ yU_0(x) \\ \hline T_2(x) \\ yU_1(x) \\ \hline T_3(x) \\ yU_2(x) \\ \hline T_4(x) \\ yU_3(x) \\ \hline \vdots \end{array}
\equiv
\begin{array}{c} 1 \\ \hline \cos\theta \\ \sin\theta \\ \hline \cos 2\theta \\ \sin 2\theta \\ \hline \cos 3\theta \\ \sin 3\theta \\ \hline \cos 4\theta \\ \sin 4\theta \\ \hline \vdots \end{array}
$$

OPs on the circle

- Not to be confused with OPs on the Unit Circle (OPUC) a la Simon, which are polynomials in $z = x + \mathrm{i}y$
  - OPUC concerns spectral theory of orthogonal operators, where here we have commuting symmetric operators

- Consider weights with the symmetry $w(x,y) = w(x,-y)$
- We can write the inner product as

$$\langle f, g \rangle = \int_{-1}^{1} \left[f(x, \sqrt{1-x^2}) g(x, \sqrt{1-x^2}) + f(x, \sqrt{1-x^2}) g(x, -\sqrt{1-x^2}) \right] w(x) \, \mathrm{d}x$$

- Define two weights on $[-1, 1]$:

$$w_p(t) = \frac{w(t)}{\sqrt{1-t^2}} \qquad w_q(t) = \sqrt{1-t^2} w(t)$$

and denote the corresponding OPs as $p_n(t)$ and $q_n(t)$

- A simple calculation shows that OPs on the circle are

$$\mathbb{P}_0(x,y) = p_0(x) \qquad \text{and} \qquad \mathbb{P}_n(x,y) = \begin{pmatrix} p_n(x) \\ y q_{n-1}(x) \end{pmatrix}$$

Interpolation by Arc polynomials via quadrature

OPs on the Arc

- An important special case is uniform weight on an arc $x > h$

$$w(x, y) = w(x, -y) = w(x) = \begin{cases} 1 & x > h \\ 0 & \text{otherwise} \end{cases}$$

  – Polynomials are invariant under rotations so any arc can be rotated to this canonical case

- We then get

$$\mathbb{P}_0(x, y) = 1 \qquad \text{and} \qquad \mathbb{P}_n(x, y) = \begin{cases} T_n^h(x) \\ yU_{n-1}^h(x) \end{cases}$$

  where $T_k^h(x)$ are orthogonal with respect $1/\sqrt{1-x^2}$ on $[h, 1]$ and $U_k^h(x)$ orthogonal with respect to $\sqrt{1-x^2}$ on $[h, 1]$

- We can calculate these using Stieltjes procedure / Lanczos

[Huybrechs 2010]

- In 1D, exactness of Gaussian quadrature means OPs are orthogonal with respect to a discrete inner product
  - Orthogonality w.r.t. a discrete inner product is sufficient tointerpolate by quadrature
- Let $x_1, \ldots, x_M$, $w_1, \ldots, w_M$ be the Gaussian quadrature rule associated with $1/\sqrt{1-x^2}$ on $[0, h]$
  - Recall its exact for polynomials of degree $2M - 1$
- Define the $2M$-point discrete inner product

$$\langle f, g \rangle_M = \sum_{j=1}^{M} w_j [f(x_j, y_j) + f(x_j, -y_j)]$$

where $y_j = \sqrt{1 - x_j^2}$

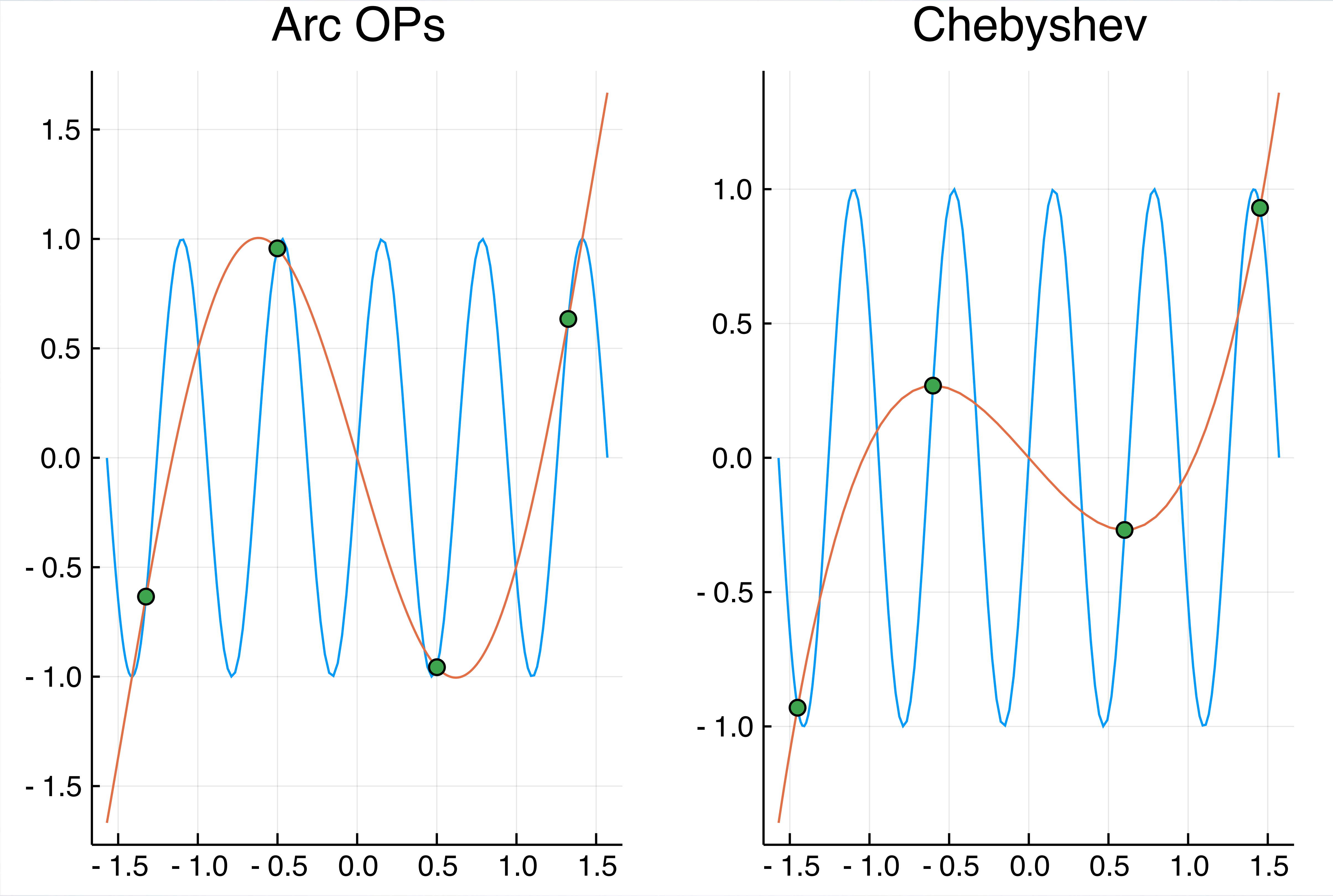
- The $2M$ polynomials $T_0^h(x), \ldots, T_{M-1}^h(x), yU_0^h(x), \ldots, yU_{M-1}^h(x)$ are orthogonal w.r.t. this discrete inner product

- The fact that orthogonality is preserved is sufficient to show that the interpolating polynomial is

$$
\begin{aligned}
f_M(x,y) = \left\langle T_0^h, f\right\rangle_M T_0^h(x) \\
&+ \sum_{n=1}^{M-1} \left[\left\langle T_n^h, f\right\rangle_M T_n^h(x) + \left\langle yU_{n-1}^h, f\right\rangle_M yU_{n-1}^h(x)\right] \\
&+ \frac{\left\langle yU_{M-1}^h, f\right\rangle_M}{\left\langle yU_{M-1}^h, yU_{M-1}^h\right\rangle_M} yU_{n-1}^h(x)
\end{aligned}
$$

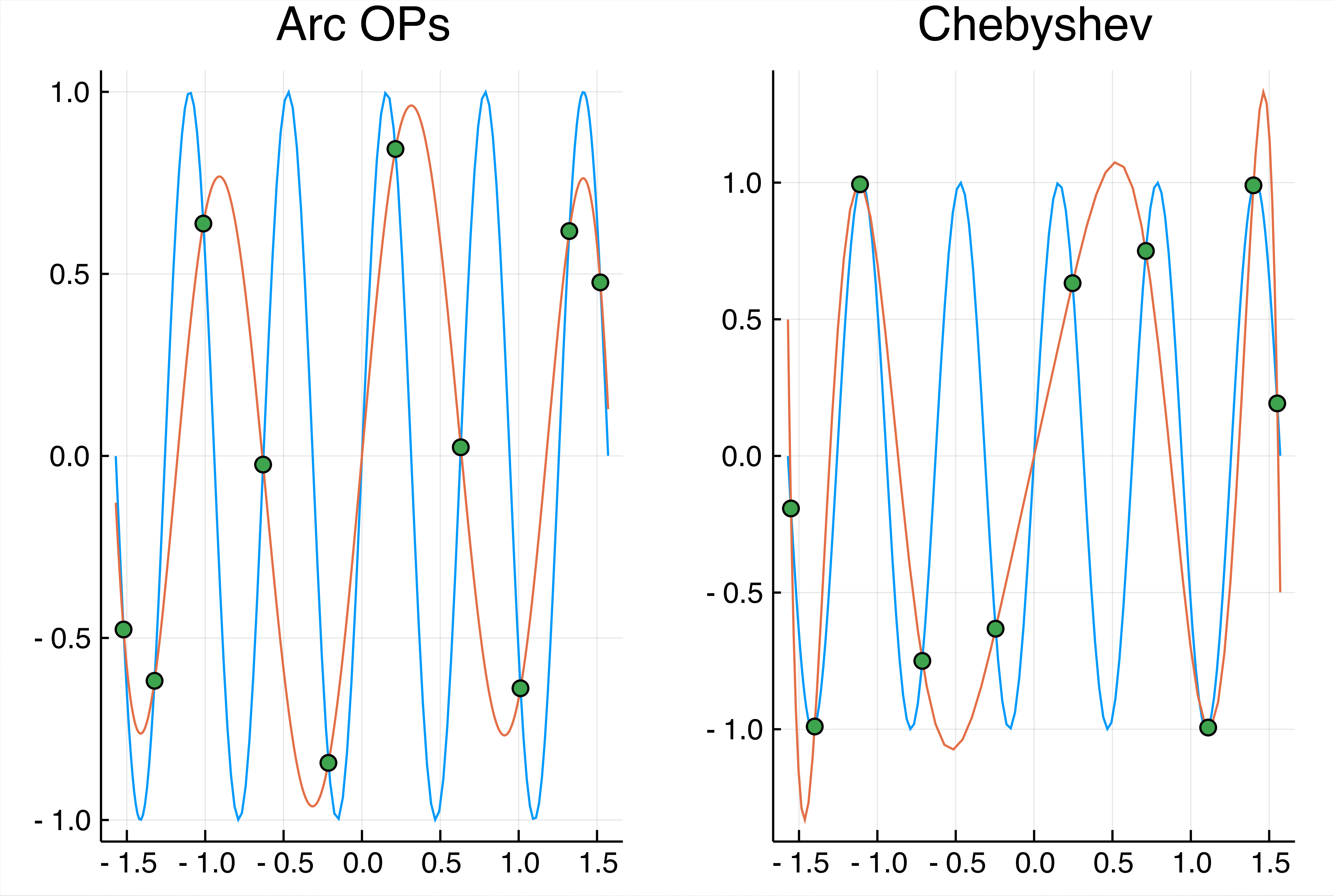
- Writing $x = \cos\theta$ and $y = \sin\theta$, this shows that we can interpolate by trigonometric polynomials on arcs
  - There is a connection with the Fourier extension problem

$$\sin 10\theta$$



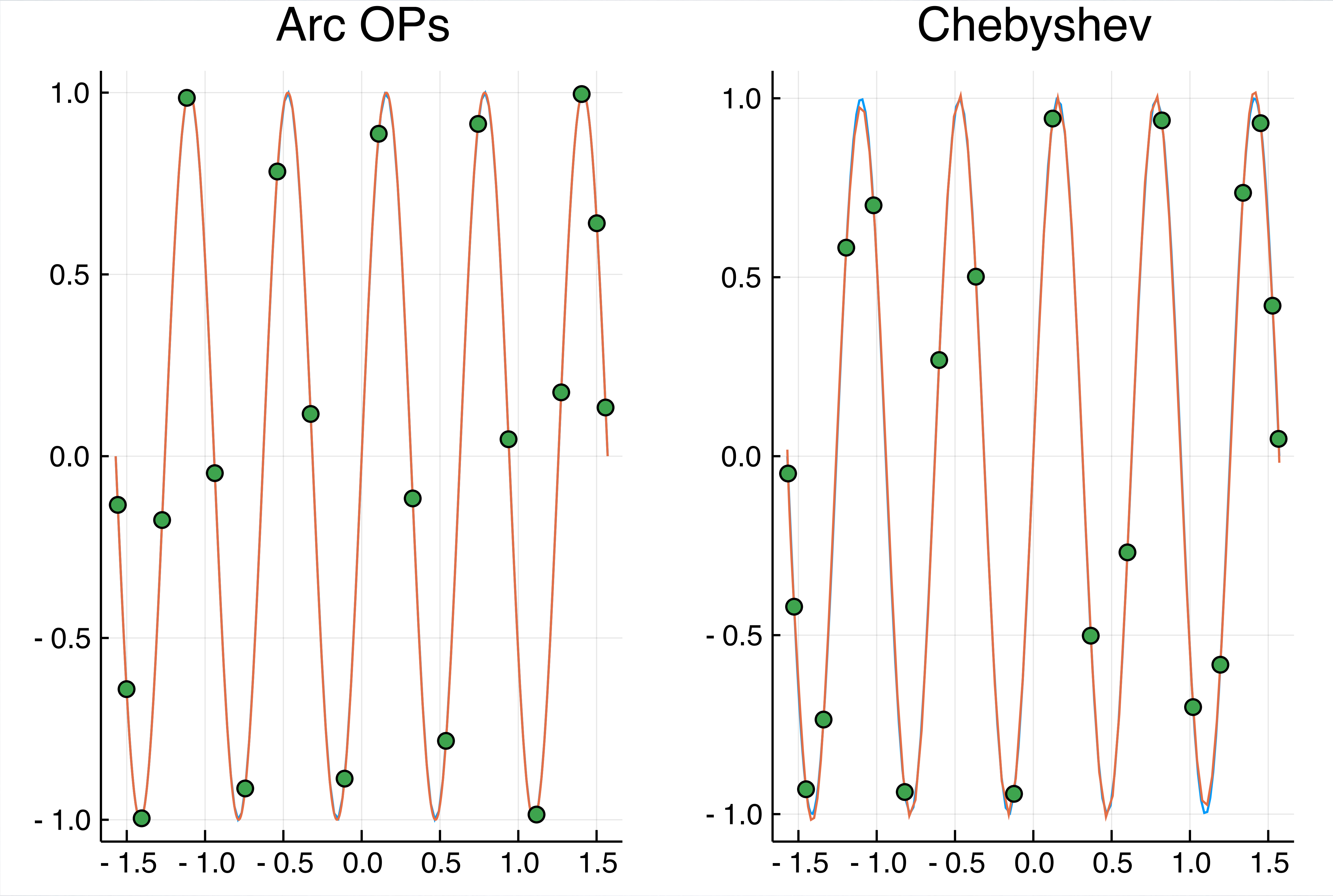
4 points

$$\sin 10\theta$$



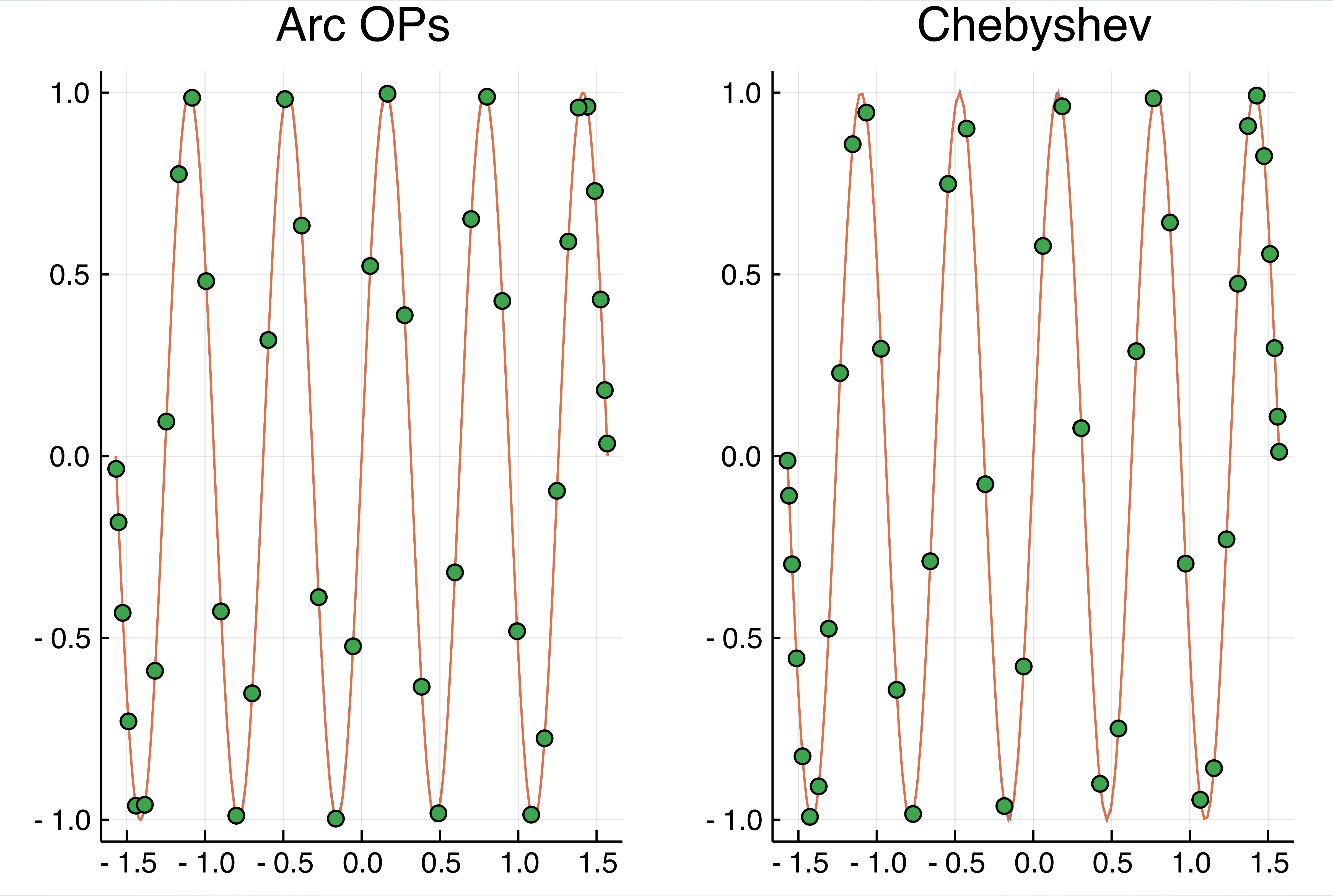
10 points

$$\sin 10\theta$$



20 points, left is exact

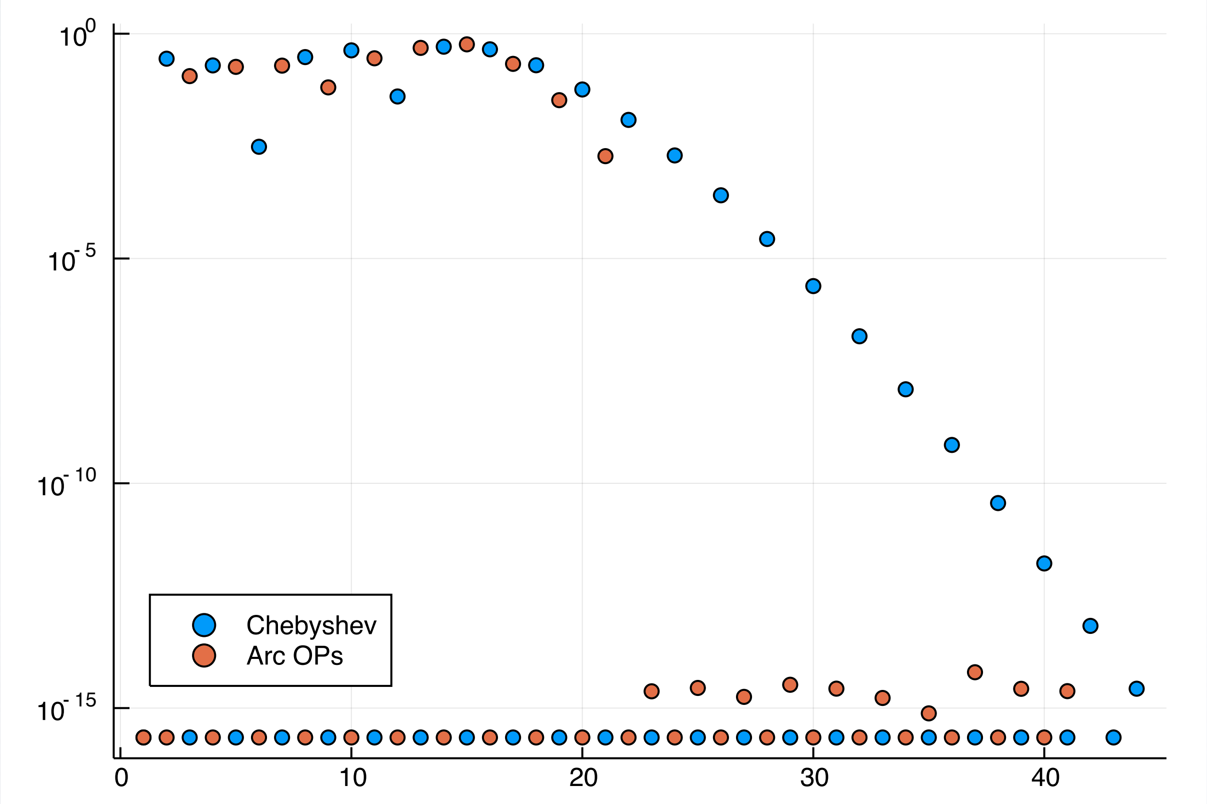
$$\sin 10\theta$$



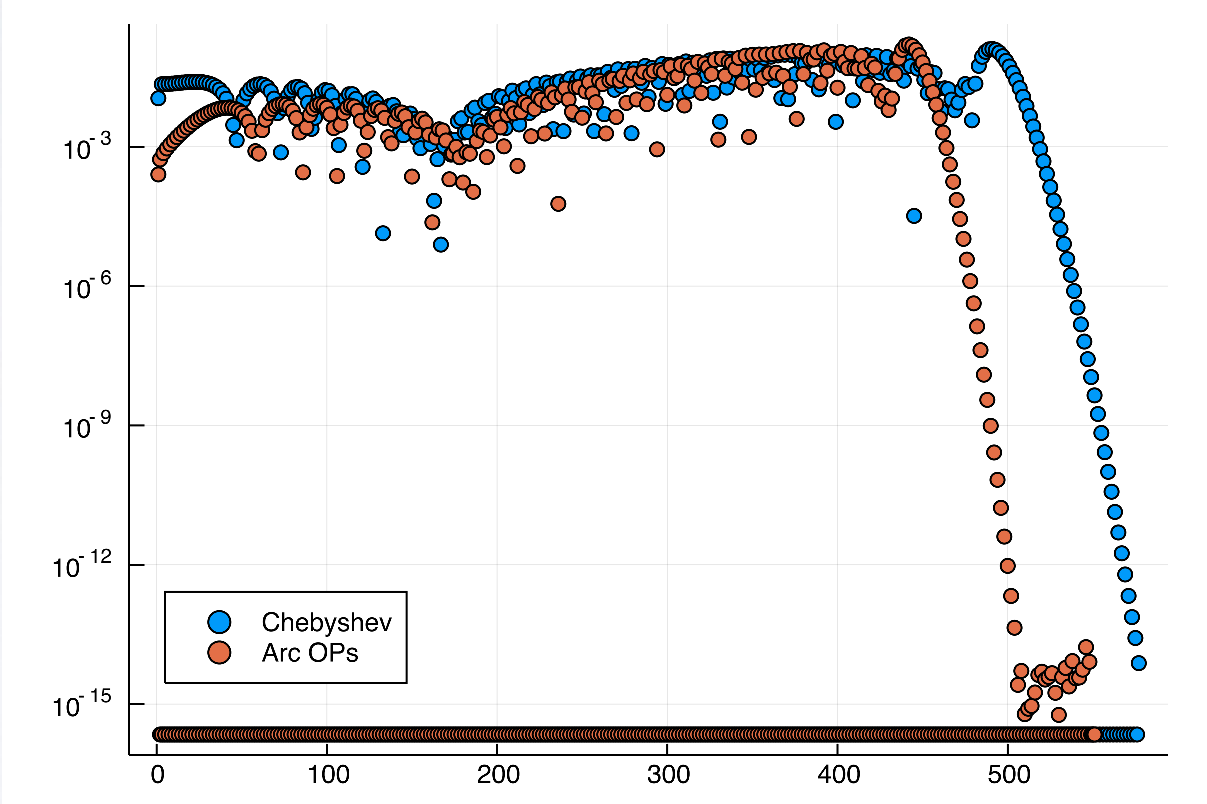
40 points, left is still exact, right is accurate to machine accuracy

Coefficient decay

$$\sin 10\theta$$



$$\left(1 + \frac{\theta^2}{\pi^2}\right) \cos \frac{10\theta}{\pi} \cos 100\theta$$



Right example from [Adcock & Huybrechs 2010]

Squares

Spheres
And
Polar caps (?)

- What about higher dimensional algebraic surfaces? Let's consider the sphere

$$x^2 + y^2 + z^2 = 1$$

- The circle gave us Fourier, the sphere gives us spherical harmonics
- Spherical harmonics have the problem of too much structure
  - Irreducible representations of $SO(3)$ (Clebsch–Gordan coefficients, ...)
  - Diagonalize the spherical Laplacian
  - Diagonalize the integral operator $1/\|\boldsymbol{x} - \boldsymbol{y}\|$
  - Millions of papers by physics, representation theorists, computational mathematicians, weather modellers...
- Idea: forget this all and treat them as *orthogonal polynomials* in x, y, z
- Extends to polar caps!

Spherical harmonics
(not typical basis)

$$P_{n,k}^{(-1/2)}(x, y), zP_{n,k}^{(1/2)}(x, y)$$

OPs with respect to $(1 - x^2 - y^2)^\mu$ on disk

Spherical harmonics
(not typical basis)

$$P_{n,k}^{(-1/2)}(x, y), zP_{n,k}^{(1/2)}(x, y)$$

OPs with respect to $(1 - x^2 - y^2)^\mu$ on disk

Polar cap OPs

$$H_{n,k}^{(-1/2)}(x, y), zH_{n,k}^{(1/2)}(x, y)$$

OPs with respect to $(1 - x^2 - y^2)^\mu$ on disk slice

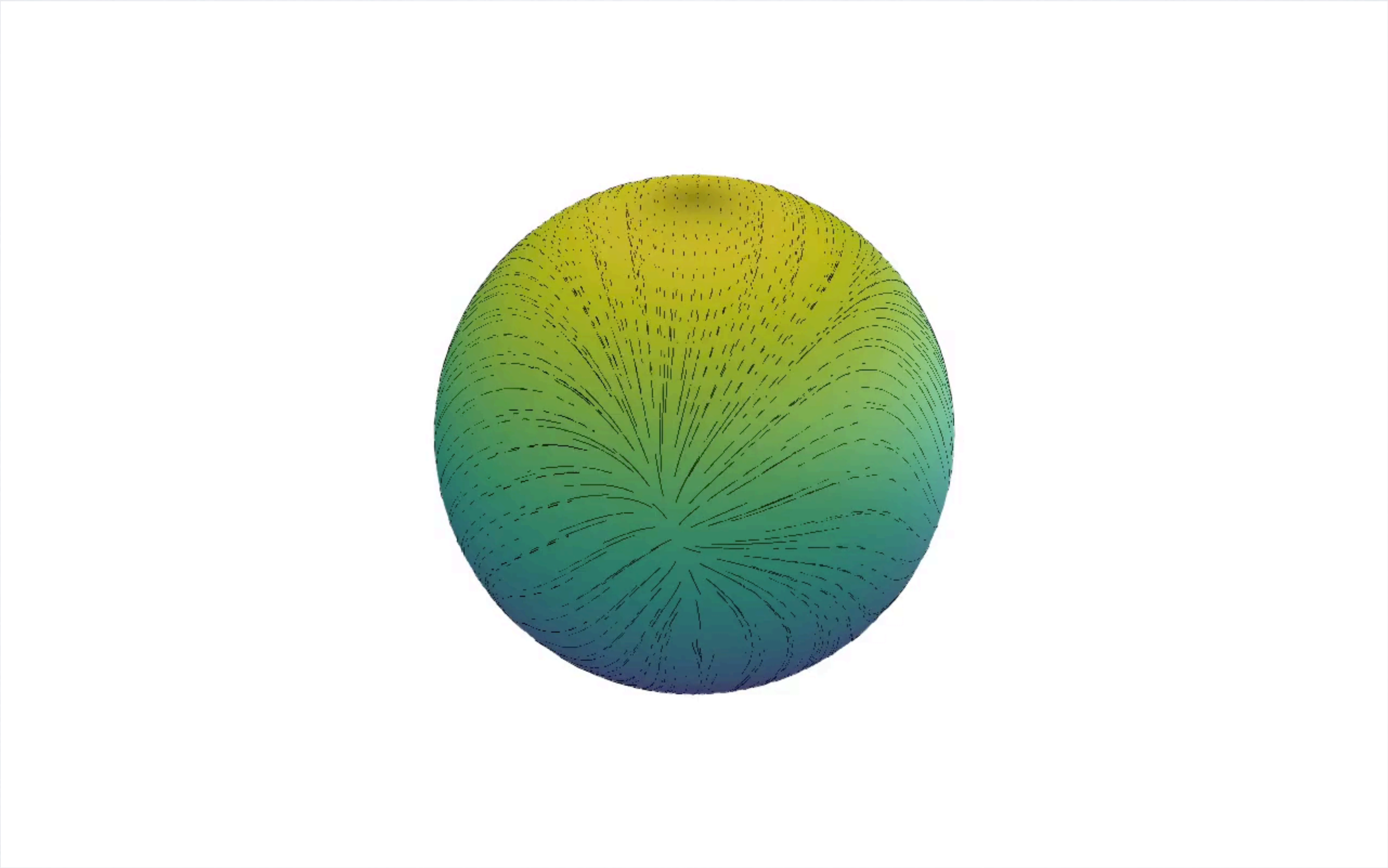
- Shallow water equation with coriolis force:

$$\mathbf{u}_t = -f(x, y, z)\, \mathbf{n} \times \mathbf{u} + \nabla h = 0$$

$$h_t = -H \nabla \cdot \mathbf{u}$$

  – $h(x, y, z)$ is expanded in spherical harmonics, $\mathbf{u}$ in the tangent space, and $\mathbf{n} = (x, y, z)^\top$ is the unit normal

  – H is the reference height and $f(x, y, z) = \frac{4\pi}{T} z$ where T is the length of 1 day in seconds



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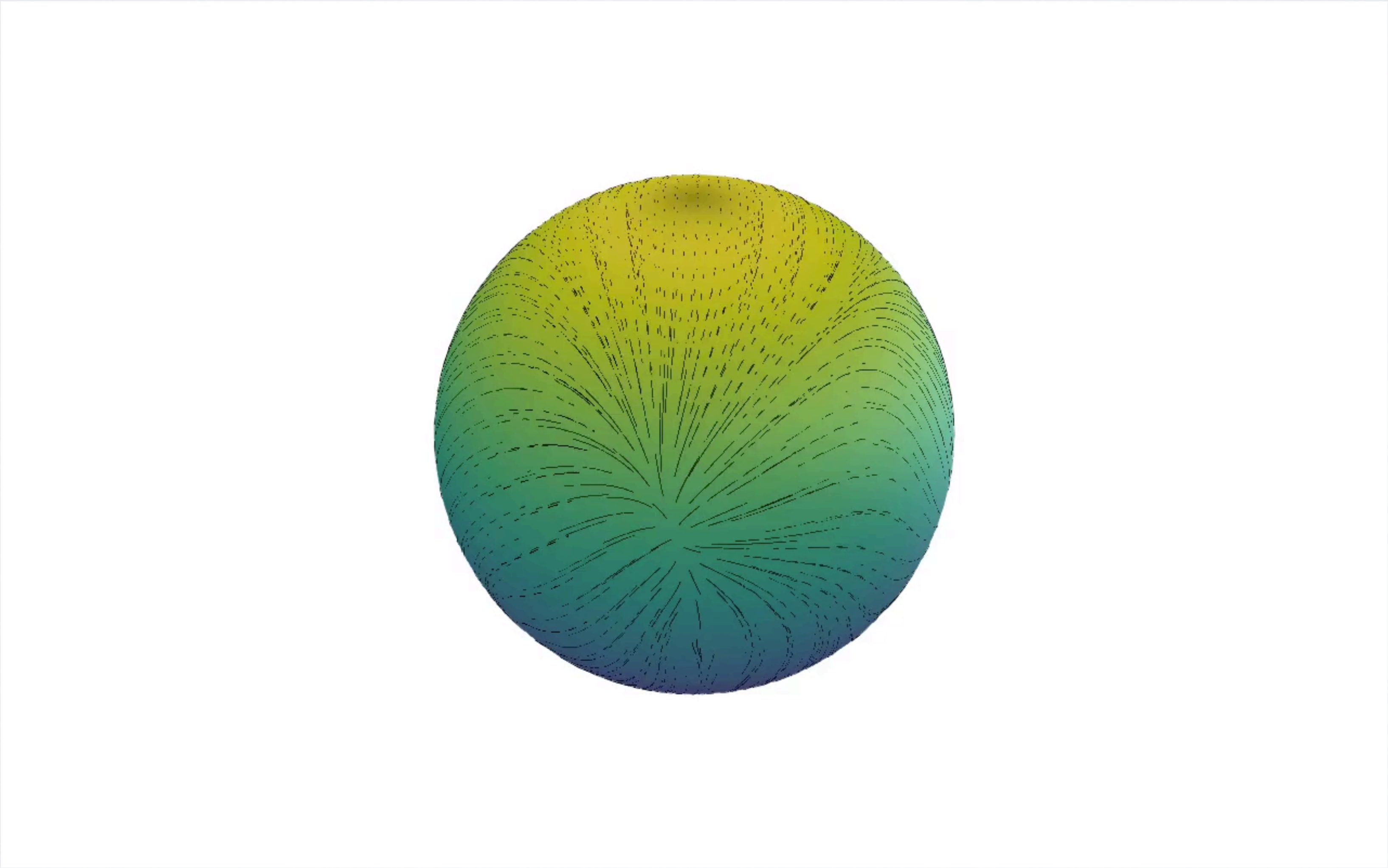


Quadratic Curves

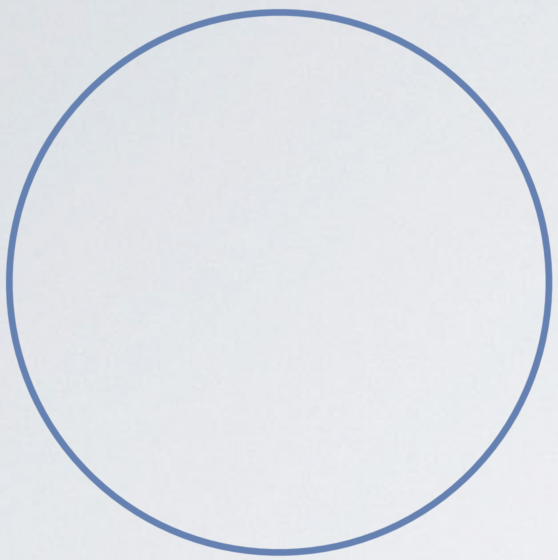
- The circle is a bit boring, so lets push this idea further and consider general quadratic curves, that is, roots to polynomials of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

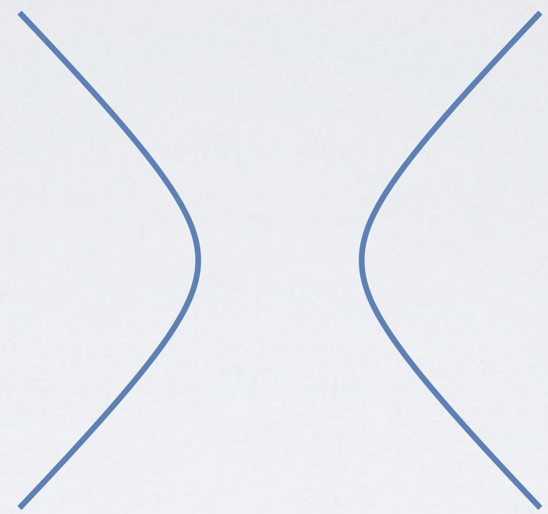
- By affine transformations, we can reduce this to 5 canonical examples (circles, hyperbolas, parabolas, crosses, and parallel lines)

- Just as in the circle, in each case we have a collapse in dimension, so that the degree $n \geq 1$ polynomials are of dimension 2

5 Canonical Cases

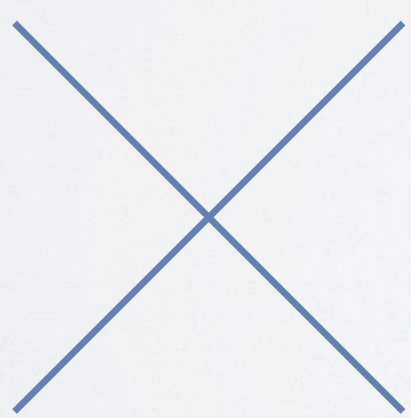


$$x^2 + y^2 = 1$$



$$x^2 = y^2 + 1$$

$$y = x^2$$

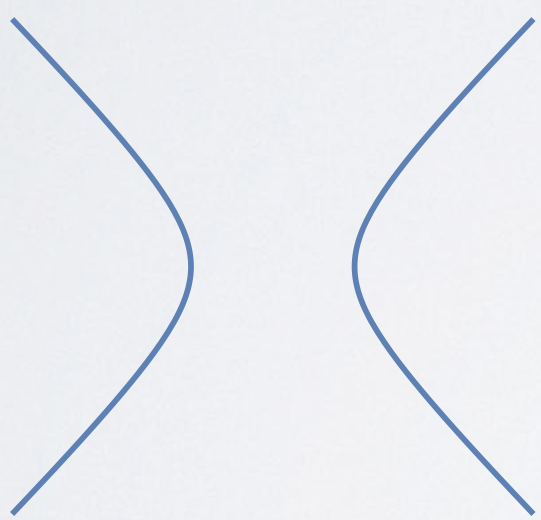


$$x^2 = y^2$$

$$y^2 = 1$$

- For each of these 5 cases, we can on a case-by-case basis reduce the problem to two families of 1D OPs, for weights with suitable symmetric properties
    - And in each of the 5 cases, we can construct an interpolative quadrature rule
- We consider only two cases: Hyperbola on one or two branches

Two branches



$$w(x,y) = w(-x,y) = w(y)$$

One branch

$$w(x,y) = w(x,-y) = w(x)$$

OPs on one branch Hyperbola

- Consider weights of the form

$$w(x, y) = w(x, -y)$$

  supported on (possibly a subset of) $x \geq 1$

- We can write the inner product as

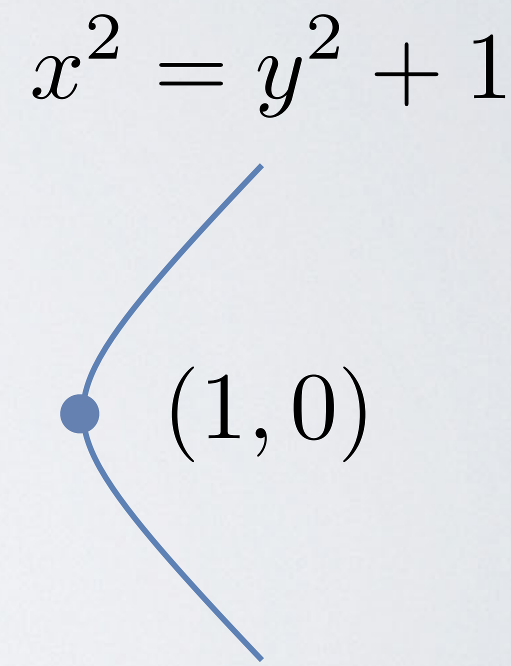
$$\int_{x^2=y^2+1} f(x,y)g(x,y)w(x,y)\,\mathrm{d}\boldsymbol{s} =$$

$$\int_1^\infty \left[f(x,\sqrt{x^2-1})g(x,\sqrt{x^2-1}) + f(x,\sqrt{x^2-1})g(x,-\sqrt{x^2-1})\right] w_0(x)\,\mathrm{d}x$$

- Let $p_n(t)$ denote OPs with respect to $w_0(t)$ and $q_n(t)$ denote OPs with respect to $w_1(t) = (t^2-1)w(t)$

- OPs on the hyperbola are then

$$\mathbb{P}_0(x,y) = p_0(x) \qquad \text{and} \qquad \mathbb{P}_n(x,y) = \begin{pmatrix} p_n(x) \\ y q_{n-1}(x) \end{pmatrix}$$



Application: Interpolation of Nearly singular functions

- Consider a function on the interval $[-1, 1]$ of the form

$$f(t) = f(t, \sqrt{t^2 + \epsilon^2})$$

where $f(x, y)$ is smooth in x and y on the hyperbola $x^2 = y^2 + \epsilon^2$

- As an example

$$f(t) = \sin(10t + 20\sqrt{t^2 + \epsilon^2})$$

becomes a "nice" function

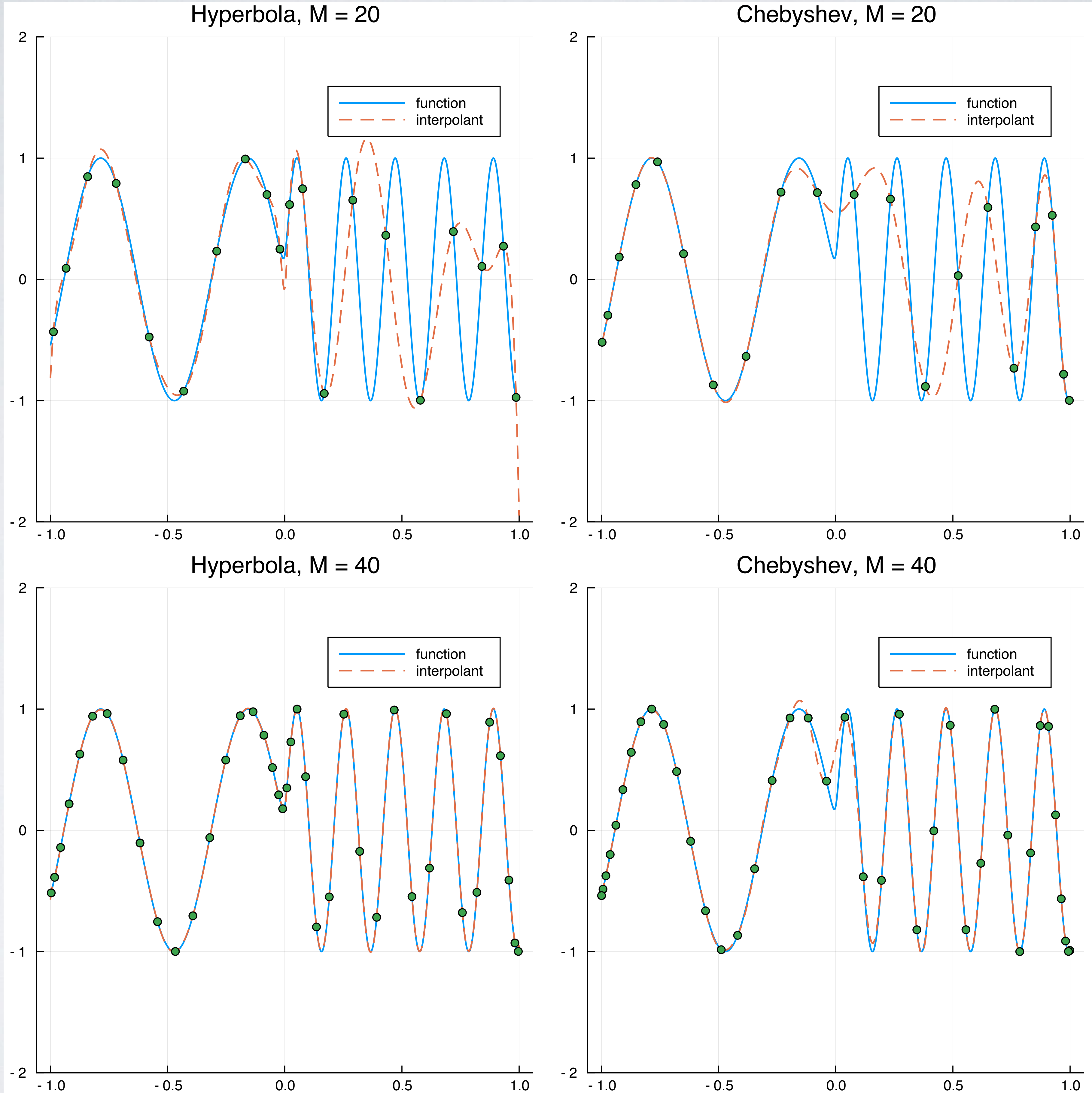
$$f(x, y) = \sin(10x + 20y)$$

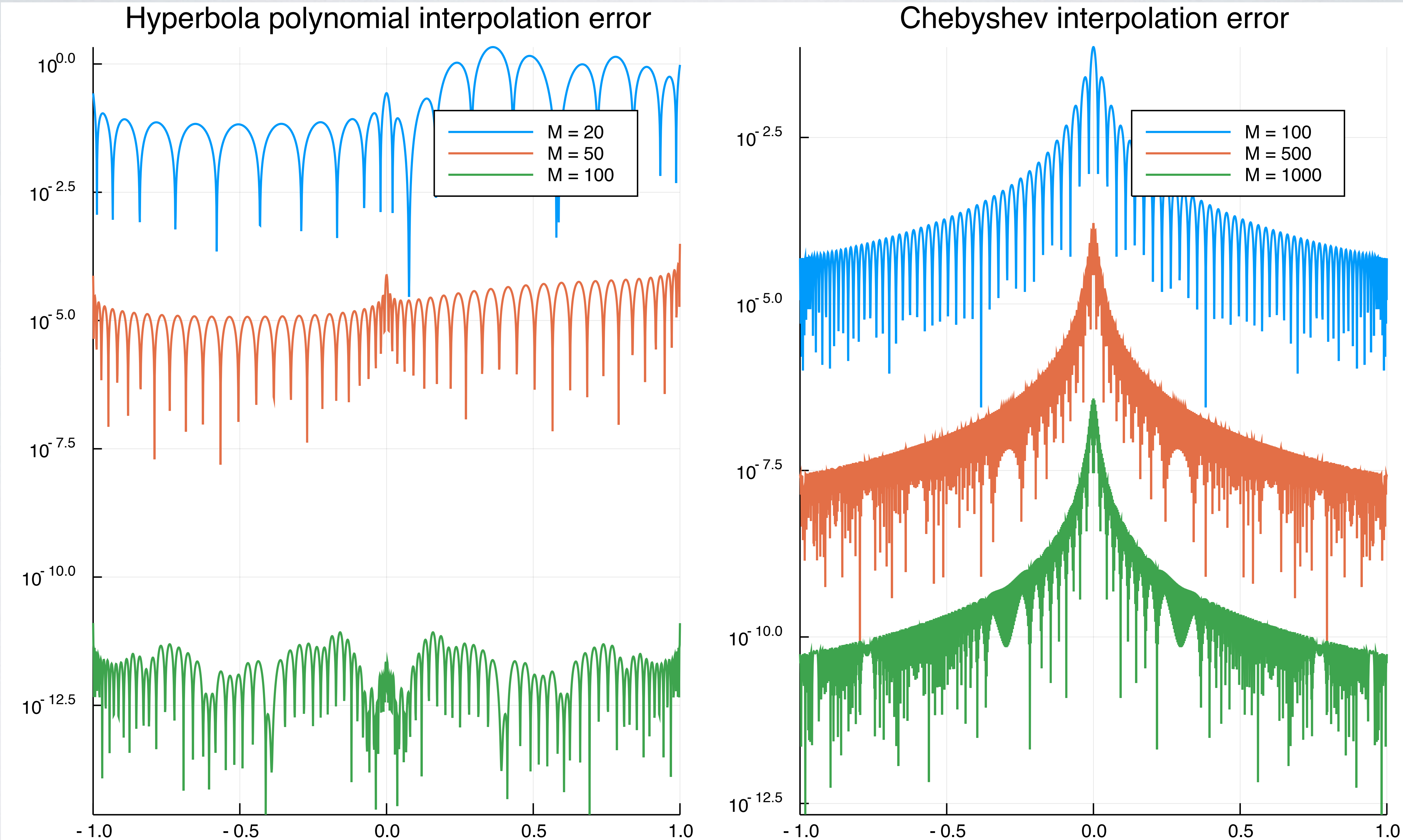
- Idea: interpolate $f(x, y)$ by $f_M(x, y)$ using OPs on the hyperbola at the points (x_j, y_j) coming from Gaussian quadrature from w_0 so that

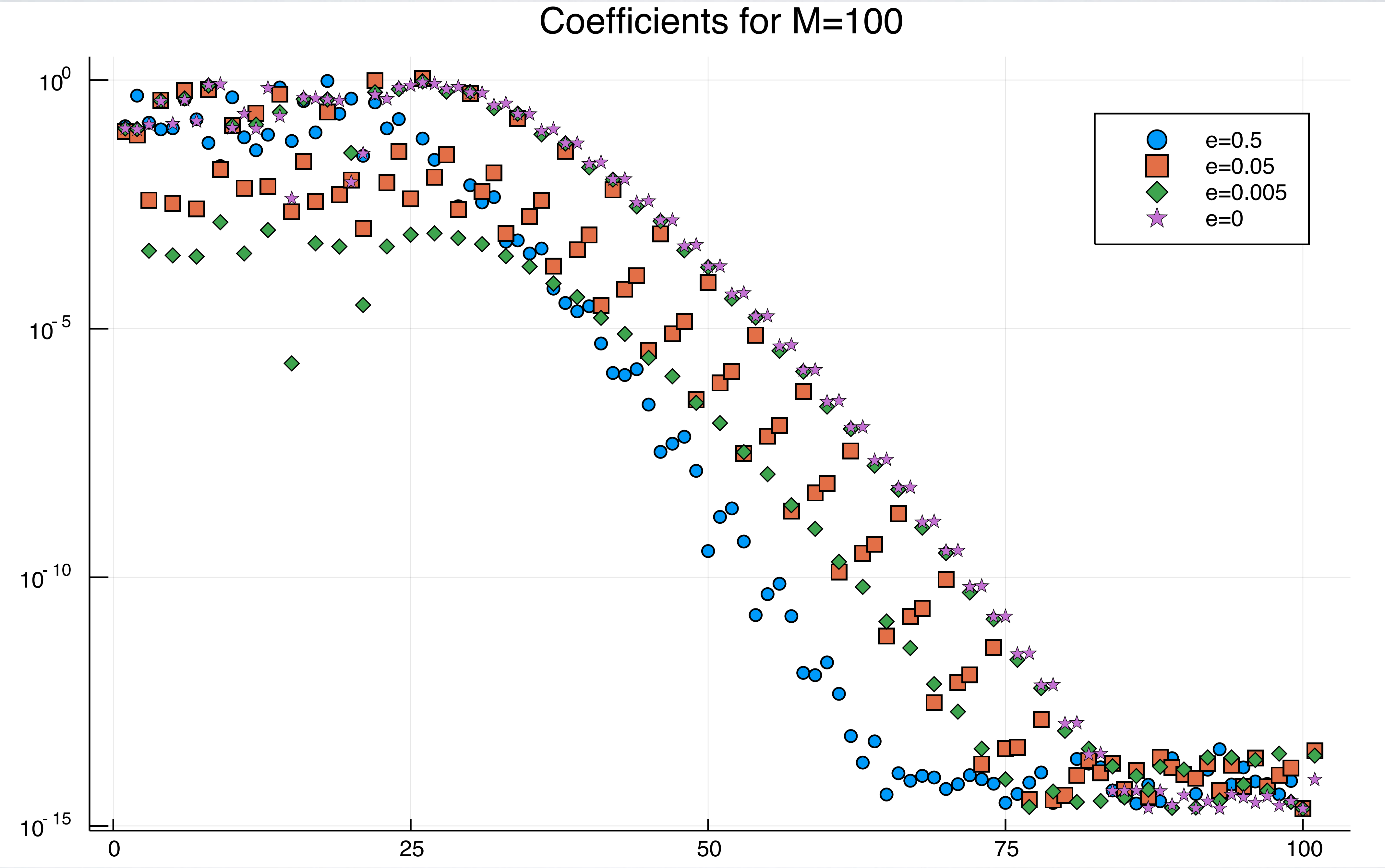
$$f_M(t) = f_M(t, \sqrt{t^2 + \epsilon^2})$$

interpolates at the points x_j

- Just like the arc, the interpolation coefficients come from quadrature


Hyperbola, M = 20
Chebyshev, M = 20
Hyperbola, M = 40
Chebyshev, M = 40
function
interpolant
2
1
0
-1
-2
-1.0
-0.5
0.0
0.5
1.0

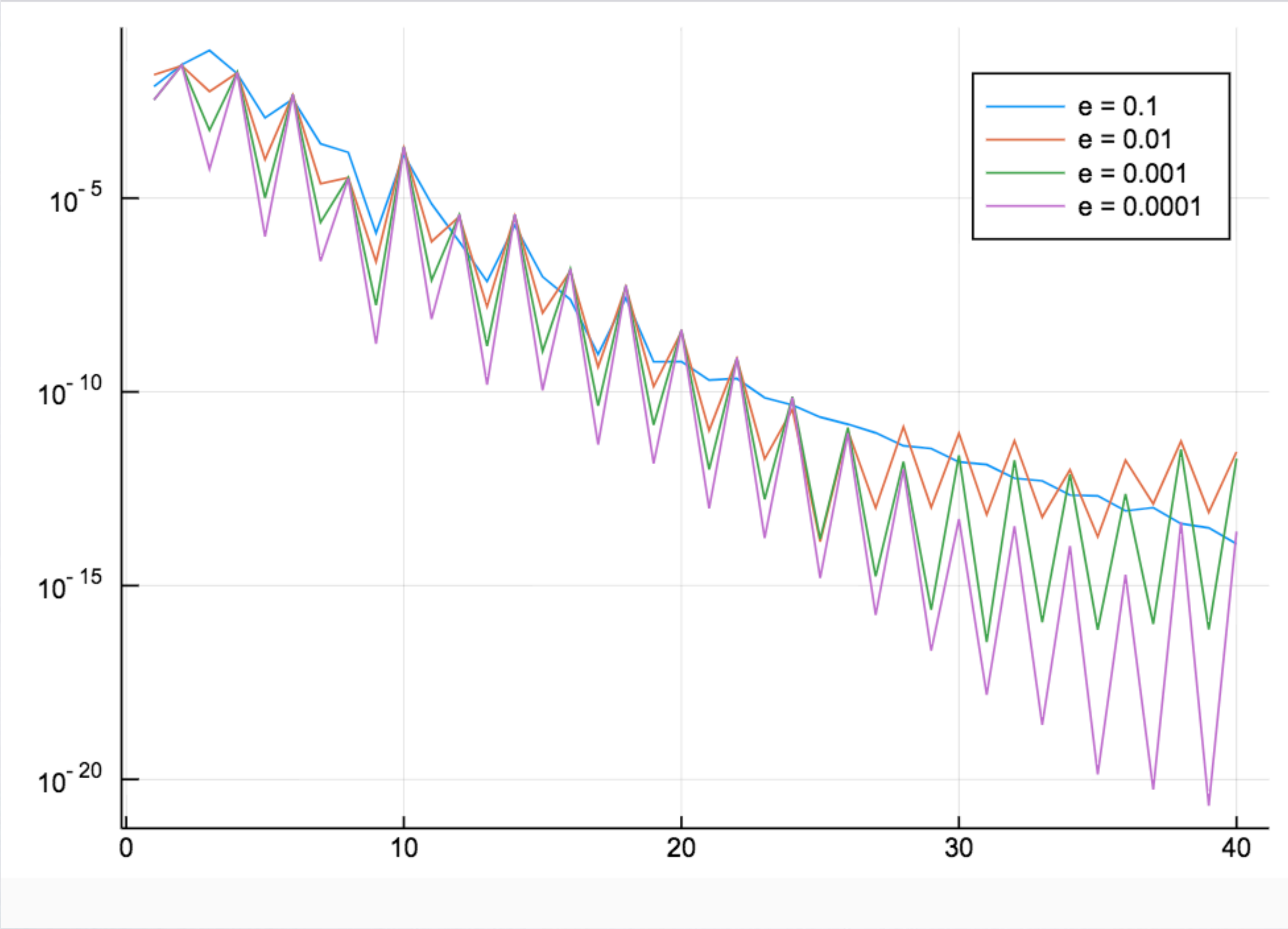

Hyperbola polynomial interpolation error
$10^{0.0}$
$10^{-2.5}$
$10^{-5.0}$
$10^{-7.5}$
$10^{-10.0}$
$10^{-12.5}$
M = 20
M = 50
M = 100
-1.0
-0.5
0.0
0.5
1.0
Chebyshev interpolation error
$10^{-2.5}$
$10^{-5.0}$
$10^{-7.5}$
$10^{-10.0}$
$10^{-12.5}$
M = 100
M = 500
M = 1000
-1.0
-0.5
0.0
0.5
1.0



Coefficients for M=100
10^0
10^-5
10^-10
10^-15
0
25
50
75
100
e=0.5
e=0.05
e=0.005
e=0

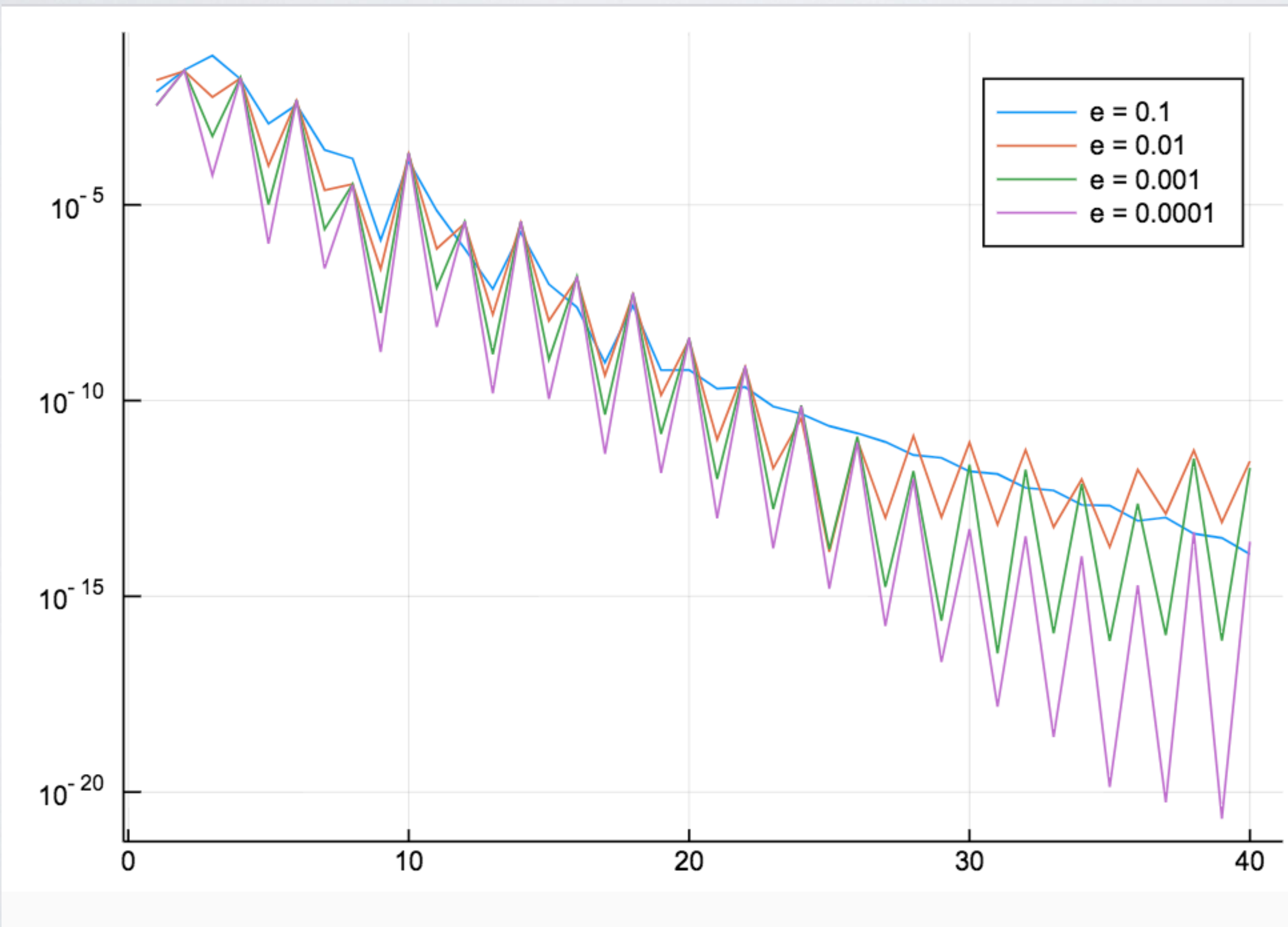
Collocation Coefficients

$$u' + \sin(t + \sqrt{t^2 + \epsilon^2})u = 0, \qquad u(-1) = 1$$



Collocation Coefficients

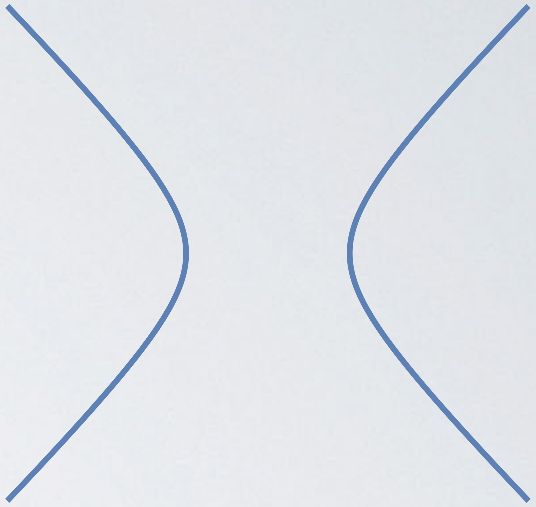
$$u' + \sin(t + \sqrt{t^2 + \epsilon^2})u = 0, \qquad u(-1) = 1$$



Would require
16k Chebyshev
coefficients
to even resolve
variable coefficient

OPs on two branch Hyperbola

$$x^2 = y^2 + 1$$



- Consider weights of the form

$$w(-x, y) = w(x, y)$$

- We can write the inner product as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \Big[f(\sqrt{y^2+1}, y) g(\sqrt{y^2+1}, y) + f(-\sqrt{y^2+1}, y) g(-\sqrt{y^2+1}, y) \Big] w(y) dy$$

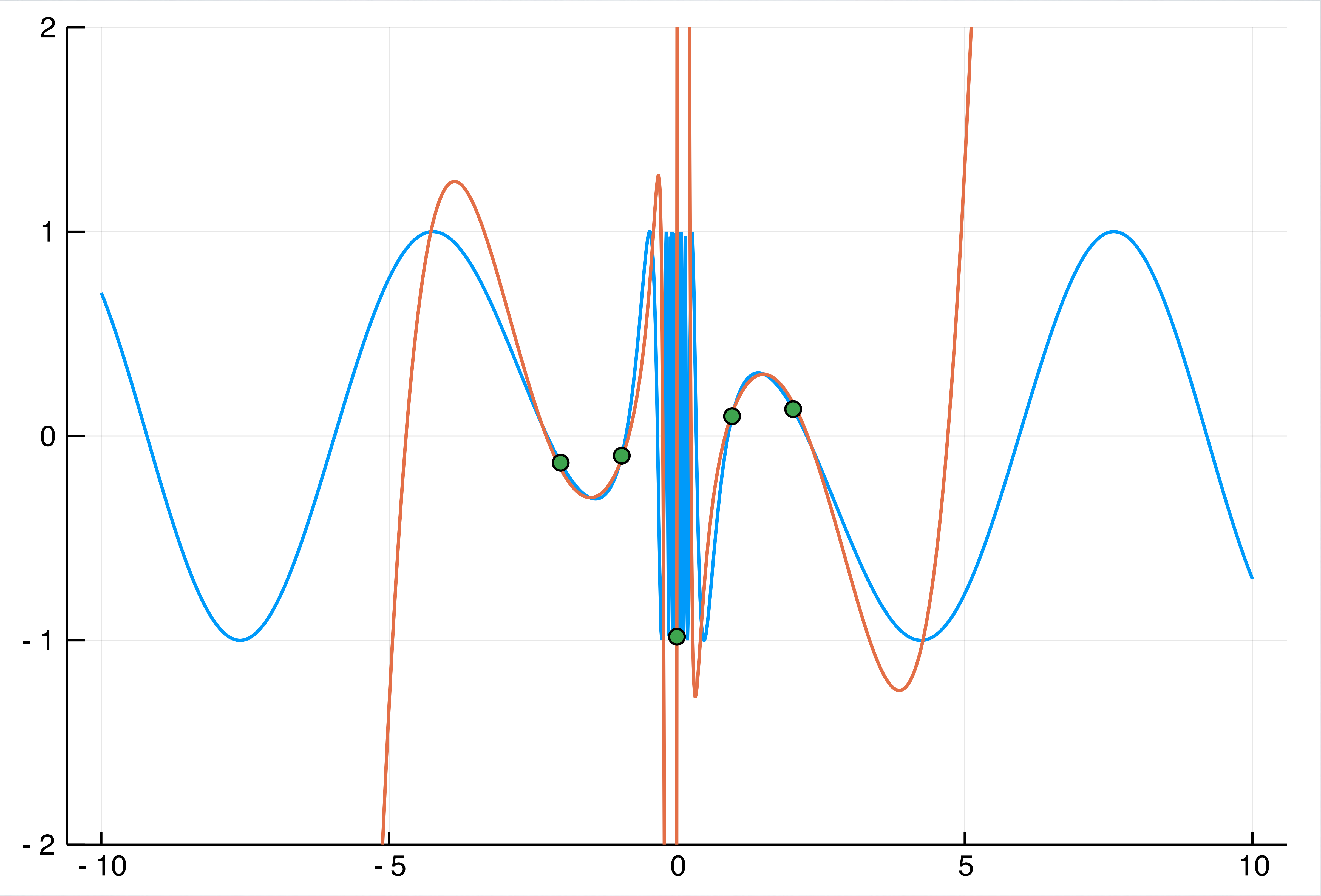
- Let $p_n(t)$ denote OPs with respect to $w(t)$ and $q_n(t)$ denote OPs with respect to $w_1(t) = (1 + t^2) w(t)$

- OPs on the hyperbola are then

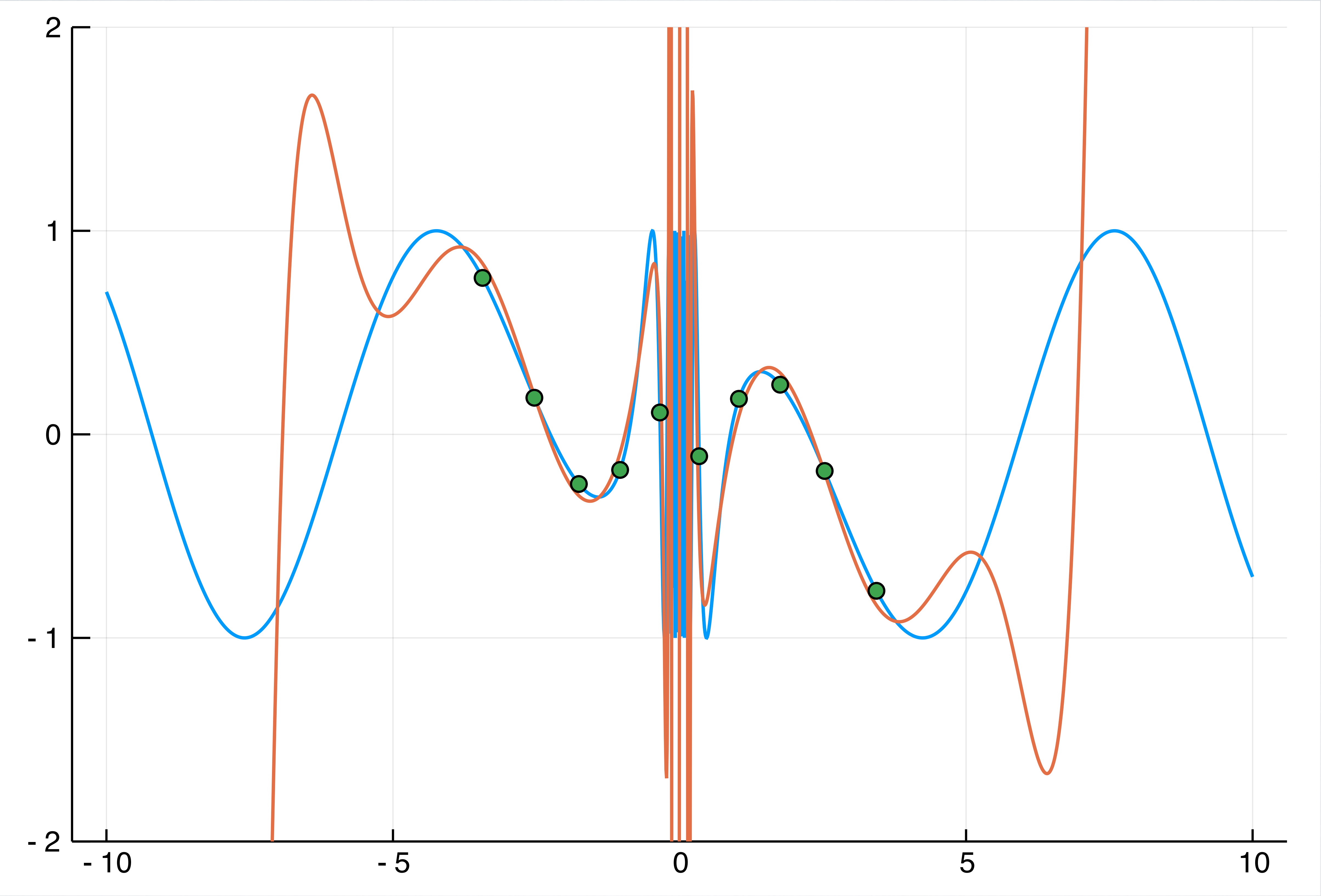
$$\mathbb{P}_0(x, y) = p_0(y) \qquad \text{and} \qquad \mathbb{P}_n(x, y) = \begin{pmatrix} p_n(y) \\ x q_{n-1}(y) \end{pmatrix}$$

Application:
Interpolation
Of
Functions with pole singularities

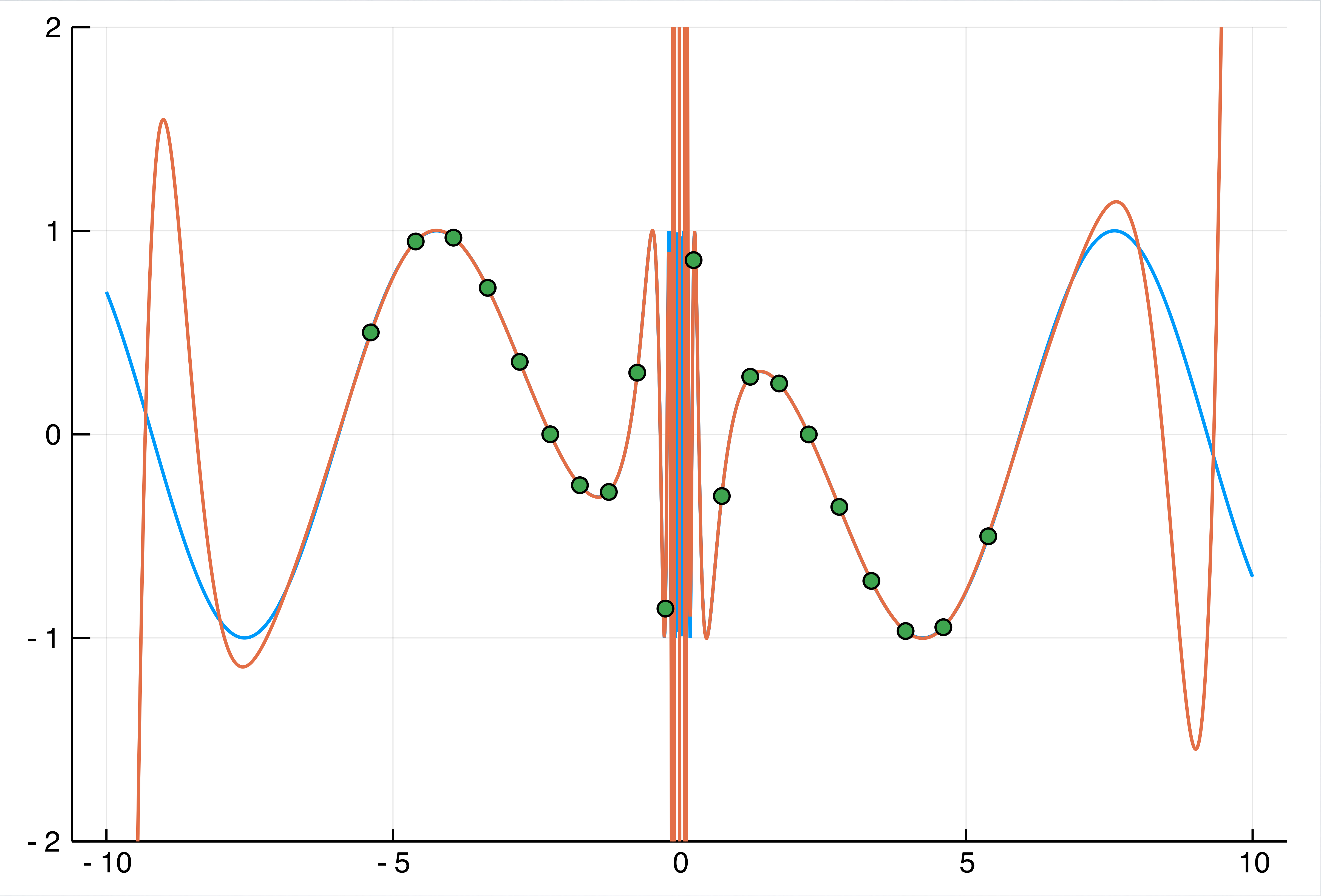
- Consider
$$f(t) = \sin(t + 2/t)$$
on the real line
- Project f to the two-branch hyperbola $x^2 = y^2 + 1$ as
$$f(x, y) = f(x - y) = \sin\left(x - y + 2(x + y)\right)$$
  - Note $t = x - y$ then $t^{-1} = x + y$
- We use Gaussian weight $w(t) = \mathrm{e}^{-t^2}$ with Gauss–Hermite points, calculated with high precision arithmetic (`BigFloat`)
  - Again, $w_1(t) = (1 + t^2)w(t)$ is non-classical so we use Stieltjes procedure with high precision arithmetic



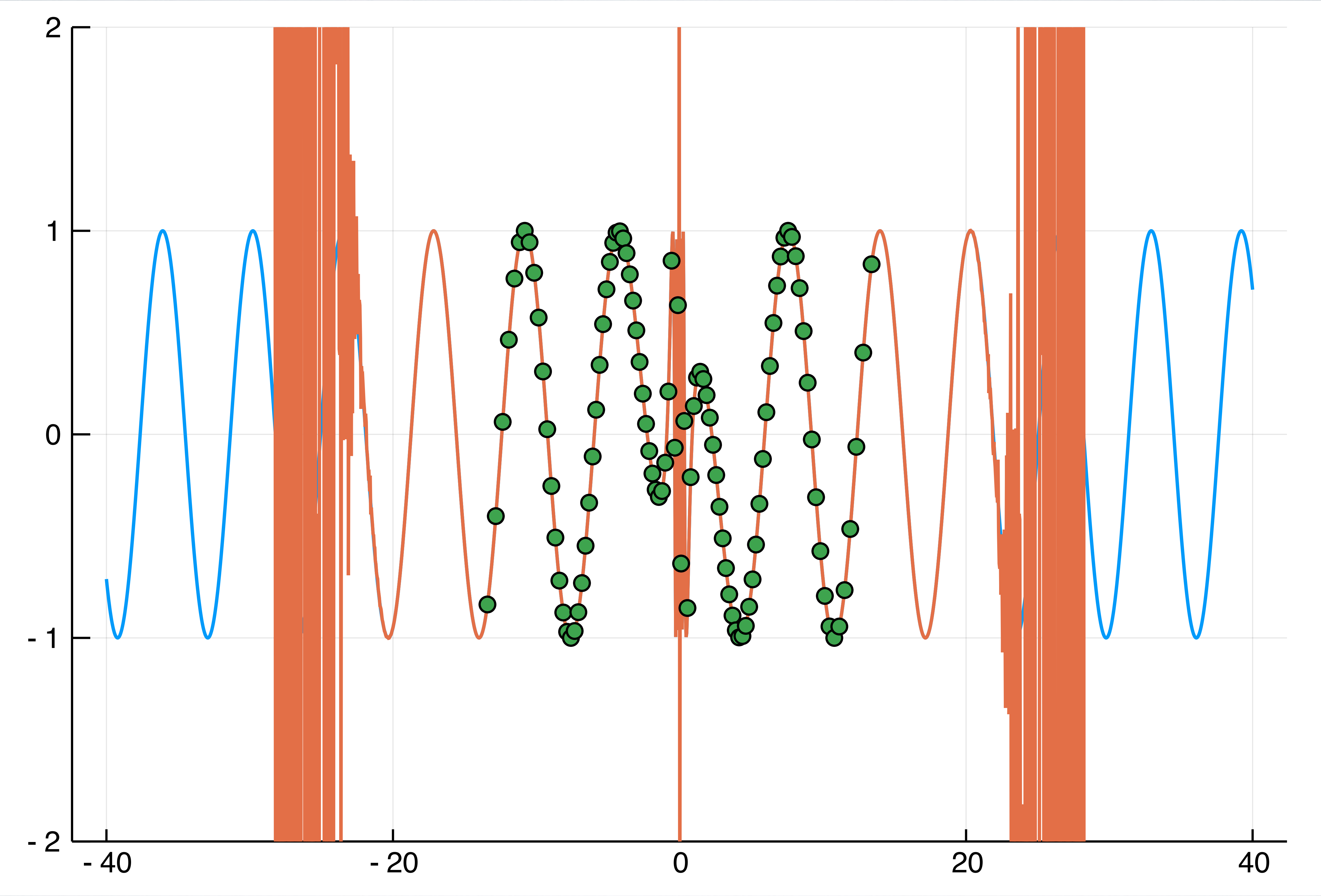
10 points



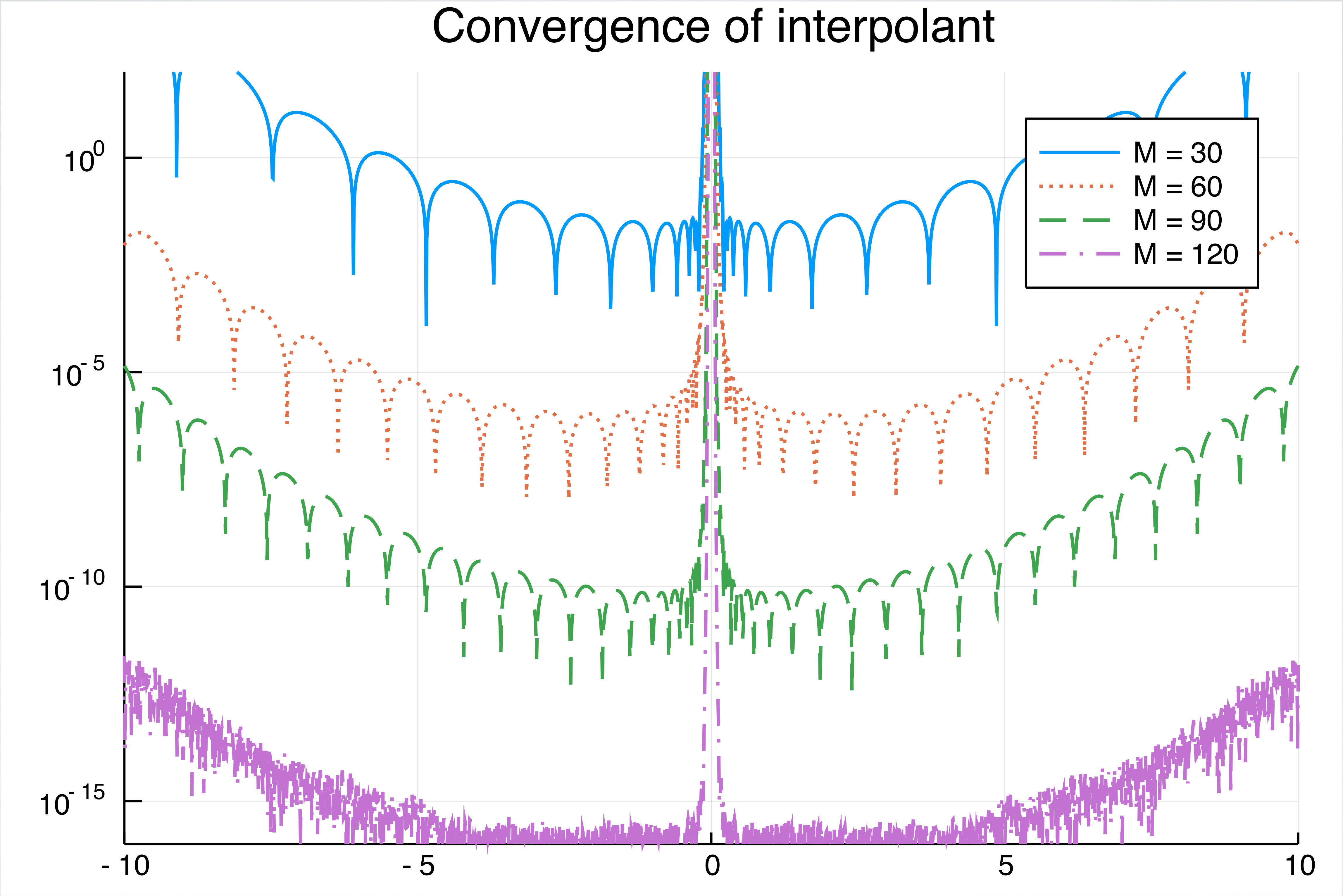
20 points



40 points



200 points


Convergence of interpolant
M = 30
M = 60
M = 90
M = 120
10^{0}
10^{-5}
10^{-10}
10^{-15}
-10
-5
0
5
10

SQUARES

- Consider solving PDEs on the square with Dirichlet conditions, using a polynomial basis
- The restriction operator maps polynomials inside the square to polynomials on the boundary of a square
- This is also an (4th order) algebraic curve:

$$(1 - x^2)(1 - y^2) = 0$$

What does the space of polynomials look like?

1

---

x

y

---

x^2

xy

y^2

---

x^3

x^2y

xy^2

y^3

---

x^4

x^3y

$\cancel{x^2y^2 = x^2 + y^2 - 1}$

xy^3

y^4

---

x^5

x^4y

$\cancel{x^3y^2 = x^3 + xy^2 - x}$

$\cancel{x^2y^3 = x^2y + y^3 - y}$

xy^4

y^5

$\vdots$

OPs on the square

Theorem 3.2. *For $n = 0, 1, 2$, a basis for $\mathcal{BV}_n$ is denoted by $Y_{n,i}$ and given by*

$$Y_{0,1}(x,y) = 1, \quad Y_{1,1}(x,y) = x \quad Y_{1,2}(x,y) = y,$$
$$Y_{2,1}(x,y) = p_{1,1}^{\alpha,\beta,\gamma}(x^2,y^2), \quad Y_{2,2}(x,y) = xy, \quad Y_{2,3}(x,y) = p_{1,2}^{\alpha,\beta,\gamma}(x^2,y^2).$$

For $n \geq 3$, the four polynomials in $\mathcal{BV}_n^2$ that are linearly independent modulo the ideal can be given by

$$\begin{aligned}
Y_{2m,1}(x,y) &= p_{m,1}^{\alpha,\beta,\gamma}(x^2,y^2), \\
Y_{2m,2}(x,y) &= p_{m,2}^{\alpha,\beta,\gamma}(x^2,y^2), \\
Y_{2m,3}(x,y) &= xy\, p_{m-1,1}^{\alpha+1,\beta+1,\gamma}(x^2,y^2), \\
Y_{2m,4}(x,y) &= xy\, p_{m-1,2}^{\alpha+1,\beta+1,\gamma}(x^2,y^2)
\end{aligned}$$

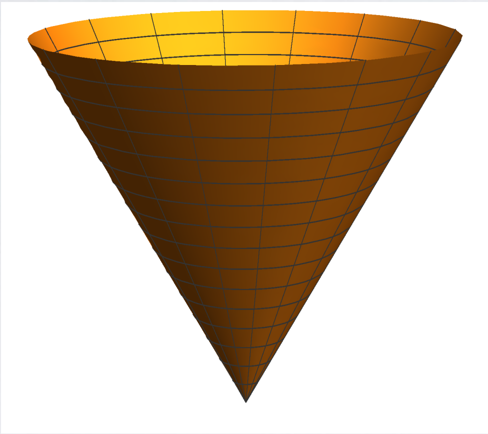
for $n = 2m \geq 2$, and

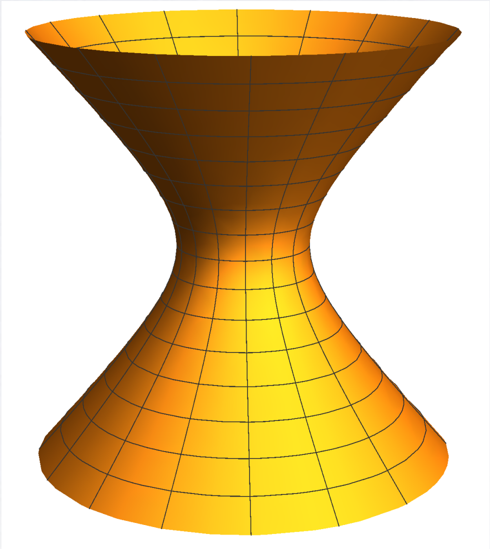
$$\begin{aligned}
Y_{2m+1,1}(x,y) &= x\, p_{m,1}^{\alpha+1,\beta,\gamma}(x^2,y^2), \\
Y_{2m+1,2}(x,y) &= x\, p_{m,2}^{\alpha+1,\beta,\gamma}(x^2,y^2), \\
Y_{2m+1,3}(x,y) &= y\, p_{m,1}^{\alpha,\beta+1,\gamma}(x^2,y^2), \\
Y_{2m+1,4}(x,y) &= y\, p_{m,2}^{\alpha,\beta+1,\gamma}(x^2,y^2)
\end{aligned}$$

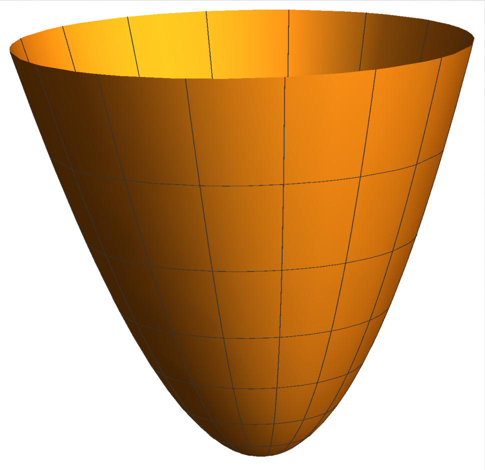
for $n = 2m+1 \geq 3$. In particular, these bases satisfy the equation $\partial_x^2 \partial_y^2 u = 0$.

Is there a good basis

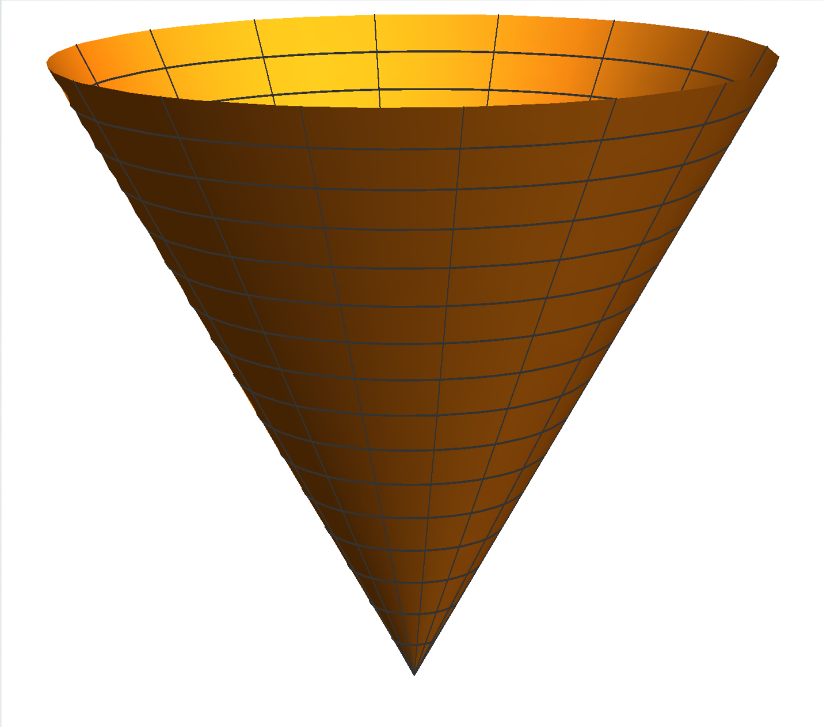
Quadratic Surfaces of Revolution







We can form OPs on and inside
quadratic curves of revolution
in arbitrary dimensions



On the cone

$$t^m P_{n-m}^{(2m+d-1,0)}(1-2t) Y_\ell^m(\mathbf{x})$$

Spherical harmonics

In the cone

$$Q_{m,\mathbf{k}}^n(\mathbf{x},t) = P_{n-m}^{(2m+d-1,0)}(1-2t) t^m P_{\mathbf{k}}^m\left(\frac{\mathbf{x}}{t}\right)$$

Ball OPs

Introducing…
ConeFun!

```
julia> @time f = Fun((t,x,y) -> exp(cos(10x*y+t))/(x^2+y^2+(t-0.1)^2), Conic());

  1.183671 seconds (3.95 M allocations: 286.165 MiB, 4.20% gc time)

julia> length(f.coefficients)
24964

julia> f(0.1, 0.1cos(0.2), 0.1sin(0.2))
269.8974361071633 4

julia> @time f = Fun((t,x,y) -> 1/(t + 0.01), Cone(), 100_000);
  1.276525 seconds (3.02 M allocations: 631.528 MiB, 3.68% gc time)
```

Uses Slevinsky's awesome FastTransforms package
which has spherical harmonic, triangular OP, and disk OP transforms

- Just like 2D, we have block-tridiagonal Jacobi operators J_x, J_y, J_z
  - In fact, the blocks are also tridiagonal (tridiagonal-block-tridiagonal)
  - And can be found in closed form via Jacobi polynomial manipulations
- Just like 2D, we can find a lower tridiagonal recurrence $L_{x,y,z}$ using pseudo-inverse of $\begin{pmatrix} B_n^x \\ B_n^y \\ B_n^z \end{pmatrix}$
  - In fact, it can be written explicitly and is $O(n)$
  - No poles!
- Just like 2D, we can use Clenshaw to construct $f(J_x, J_y, J_z)$
- We thus can reduce, e.g, variable coefficient Helmholtz

$$(\Delta_S + a(x, y, z))u = 0$$

to a banded-block-banded matrix

TANGENT SPACE OF SPHERE

- To do more complicated PDEs like shallow water, we need to work with $\boldsymbol{u}$ in the tangent space of the sphere
- We will represent these using vector-valued polynomials $p(x, y, z)$ restricted to the sphere in the ideal $\boldsymbol{n}(x, y, z) \cdot \boldsymbol{p}(x, y, z) = 0$
  - Here $\boldsymbol{n}(x, y, z) = (x, y, z)^\top$ is the unit normal
- Sounds complicated... but turns out using the surface gradient of spherical harmonics

$$\nabla_S \mathbb{P}_n \qquad \text{and} \qquad \boldsymbol{n} \times \nabla_S \mathbb{P}_n$$

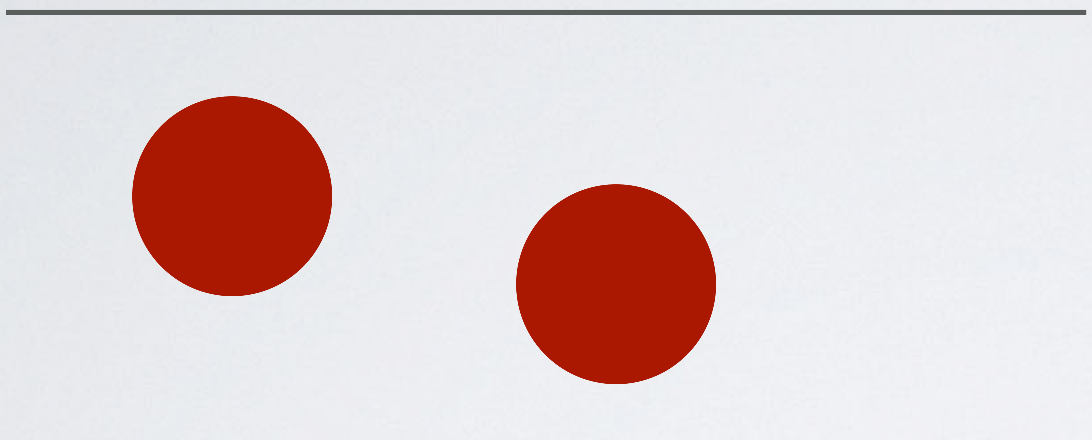
  are orthogonal and in this vector-valued polynomial space
  - And they have the structure of OPs: including tridiagonal-block-tridiagonal Jacobi operators $\boldsymbol{J}^x, \boldsymbol{J}^y, \boldsymbol{J}^z$
- We thus get banded-block-banded matrix for all of the following operators:
  - Acting on spherical harmonics: ∇_S, $f(x, y, z)$
  - Acting on tangent space: $\nabla_S \cdot$, $\boldsymbol{n}\times$, $f(x, y, z)$

Application:
Shallow water equation with Coriolis Force

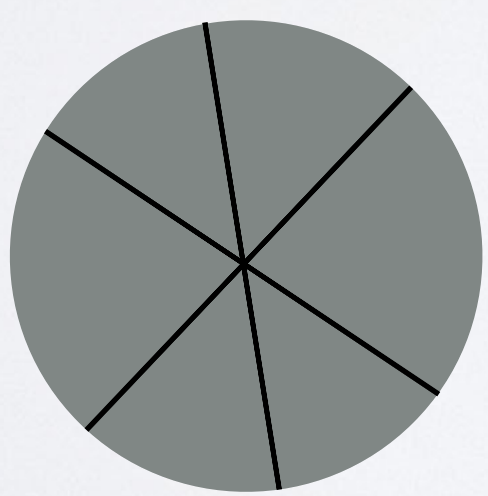
Future directions

- PDEs on spherical triangles
- PDEs in Minkowski spacetime (Hyperbolic ball)
  - Any good reason to solve with boundary conditions on the light cone??
- Sparse spectral methods for PDEs inside curves
- Other singular functions
  - $f(t) = f(t, \sqrt{t})$ on the parabola $y^2 = x$
- Boundary integral methods?

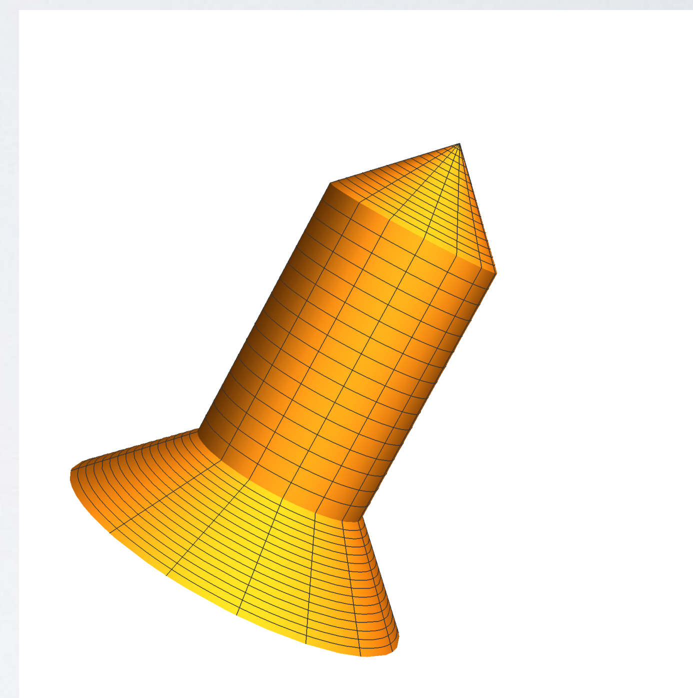
Piecewise simple geometries?



Flow in channel with obstacles



Spectral element method in pipe



Rockets!