OPs inside cubic curves

November 25, 2020

Consider for $t > 1$

$$\Omega = \{(x,y) : y^2 \leq \underbrace{(1-x^2)(t-x)}_{\rho(x)}, -1 \leq x \leq 1\}$$

Define

$$P_{nk}(x,y) := \tilde{C}_{n-k}^{(k+1)}(x)\rho(x)^{k/2}P_k(y/\rho(x))$$

where P_k are Legendre polynomials and $\tilde{C}_k^{(\lambda)}$ are monic semiclassical ultraspherical polynomials, orthogonal w.r.t. $\rho(x)^{\lambda-1/2}$. Denote its norm-squared as

$$h_k^{(\lambda)} := \int_{-1}^{1} \tilde{C}_k^{(\lambda)}(x)^2 \rho(x)^{\lambda-1/2}\mathrm{d}x$$

Note $P_{nk}(x,y)$ are orthogonal w.r.t.

$$\langle f, g\rangle := \int_{-1}^{1}\int_{-\rho(x)}^{\rho(x)} f(x,y)g(x,y)\mathrm{d}y\mathrm{d}x$$

since, with $t = y/\sqrt{\rho(x)}$ we have

$$\langle P_{nk}, P_{mj}\rangle = \int_{-1}^{1} \tilde{C}_{n-k}^{(k+1)}(x)\tilde{C}_{m-j}^{(j+1)}(x)\rho(x)^{(k+j+1)/2}\mathrm{d}x \int_{-1}^{1} P_k(t)P_j(t)\mathrm{d}t$$

Denote it's norm-squared as

$$h_{nk}^P := \langle P_{nk}, P_{nk}\rangle = h_{n-k}^{(k+1)} h_k^P$$

where h_k^P are the Legendre norms.

Let's denote the true OPs as R, and we know the first 5:

$$R_{00} := P_{00} | R_{10} := P_{10}, R_{11} := P_{11} | R_{20} := P_{20}, R_{21} := P_{21}$$

Here we use a total lexicographical ordering of polynomials whenever we talk about "lower order terms".

0.1 Quadratic case

The catch: P_{nk} does not include all polynomials, so for example $R_{22} \neq P_{22}$ since

$$P_{22}(x,y) = \rho(x)P_2(y/\rho(x)) = \frac{3}{2}y^2 - \frac{\rho(x)}{2} = -\frac{x^3}{2} + \text{l.o.t}$$

which is cubic in x. Idea is to use

$$P_{30}(x,y) = \tilde{C}_3^{(1)}(x) = x^3 + O(x^2)$$

to kill of the cubic term via

$$R_{22} := P_{22} + \frac{P_{30}}{2} = \frac{3}{2}y^2 + \text{l.o.t.}$$

Note this is indeed an OP since for lower order polynomials R_{kj} we have

$$\langle R_{22}, R_{kj}\rangle = \langle R_{22}, P_{kj}\rangle = \langle P_{22}, P_{kj}\rangle + \frac{\langle P_{30}, P_{kj}\rangle}{2} = 0.$$

If we denote

$$h_{nk}^R := \langle R_{nk}, R_{nk}\rangle$$

we have

$$h_{22}^R = h_{22}^P + \frac{h_{30}^P}{2}$$

0.2 Cubic case

The catch, $R_{30} \neq P_{30}$ since

$$\langle P_{30}, R_{22} \rangle = \frac{\langle P_{30}, P_{30} \rangle}{2} = \frac{h_{30}^P}{2} (= \frac{h_3^{(1)}}{2})$$

But the solution is simple:

$$R_{30} := P_{30} - \frac{h_{30}^P}{2h_{22}^R} R_{22}$$

Fortunately the next term is fine

$$R_{31} := P_{31}$$

since all l.o.t. R_{nk} are sums of l.o.t. P_{nk}. Now note

$$P_{32}(x,y) = \tilde{C}_1^{(2)}(x) \left(\frac{3}{2} y^2 - \frac{\rho(x)}{2} \right) = -\frac{x^4}{2} + \text{l.o.t.}$$

So

$$R_{32} := P_{32} + \frac{P_{40}}{2}.$$

Finally

$$P_{33}(x,y) = \frac{5}{2} y^3 - \frac{3}{2} \rho(x) y = -\frac{3}{2} x^3 y + \text{l.o.t.}$$

Here we use

$$P_{41}(x,y) = \tilde{C}_3^{(2)}(x) y = x^3 y + \text{l.o.t}$$

so we define

$$R_{33} := P_{33} + \frac{3}{2} P_{41}$$

0.3 Quartic case

$$R_{40} := P_{40} - \frac{h_{40}^P}{2h_{32}^R} R_{32}$$

$$R_{41} := P_{41} - \frac{3h_{41}^P}{2h_{33}^R} R_{33}$$

$$R_{42} := P_{42} + \frac{P_{50}}{2}$$

$$R_{43} := P_{43} + \frac{3}{2} P_{51}$$

The next term is the hard one as we now have two "bad" terms:

$$P_{44}(x,y) = \frac{35}{8} y^4 - \frac{30}{8} \rho(x) y^2 + \frac{3\rho(x)^2}{8} = \frac{3}{8} x^6 - \frac{3}{4} t x^5 - \frac{30}{8} x^3 y^2 + \text{l.o.t.}$$

We can clearly kill it off using P_{60}, P_{50}, P_{51} and P_{52} though it seems that the construction becomes increasingly difficult for quintic and higher.