Truncated Jacobi operators

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On the upper branch of the teardrop curve,

$$\gamma = \left\{ (x,y) : y^2 = \phi(x) := \frac{1}{4}(1-x)^2(1+x),\ y \geq 0,\ -1 \leq x \leq 1 \right\},$$

with the inner product,

$$\langle f, g\rangle = \int_{-1}^{1} fg\left(x, \sqrt{\phi(x)}\right) w_{\alpha,\beta}(x)\mathrm{d}x, \qquad w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta,$$

we do not have an explicit OP basis but we can construct it with the Gram-Schmidt procedure. The orthonormalized OP basis satisfies

$$\begin{aligned} xQ_n &= B_{n-1,1}^\intercal Q_{n-1} + A_{n,1}Q_n + B_{n,1}Q_{n+1}, \\ yQ_n &= B_{n-1,2}^\intercal Q_{n-1} + A_{n,2}Q_n + B_{n,2}Q_{n+1}. \end{aligned}$$

The Jacobi operators are asymptotically, as $n \to \infty$, block-Toeplitz with 3×3 blocks. Let $A^x = \lim_{n\to\infty} A_{n,1}$ and let A^y, B^x, B^y be similarly defined. For $\alpha = \beta = -1/2$, we find that

$$A^x = \frac{1}{8}\begin{pmatrix} -2 & -4 & -1 \\ -4 & -2 & 4 \\ -1 & 4 & -2 \end{pmatrix}, \qquad B^x = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 4 & -1 & 0 \end{pmatrix}, \tag{1}$$

and

$$A^y = v\begin{pmatrix} 12 & -1 & 6 \\ -1 & 12 & 1 \\ 6 & 1 & 12 \end{pmatrix}, \qquad B^y = v\begin{pmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 1 & 6 & 1 \end{pmatrix}, \qquad v = \frac{\sqrt{2}}{64}. \tag{2}$$

The symbols associated with the limiting x and y Jacobi operators are, respectively,

$$X(z) = \frac{(B^x)^\intercal}{z} + A^x + B^x z, \qquad Y(z) = \frac{(B^y)^\intercal}{z} + A^y + B^y z,$$

where z is on the complex unit circle. The symbols commute, satisfy the algebraic equation defining γ and the image of their joint spectrum is the support of the OPs (also γ), i.e.,

$$X(z)Y(z) = Y(z)X(z), \qquad Y(z)^2 = \phi\left[X(z)\right] = \frac{1}{4}\left[\mathrm{I} - X(z)\right]^2\left[\mathrm{I} + X(z)\right],$$

and

$$\left\{ (\lambda_{x,i}, \lambda_{y,i}) \ :\ X(z)q_i = \lambda_{x,i}q_i,\ Y(z)q_i = \lambda_{y,i}q_i,\ \lambda_{y,i} = \sqrt{\phi(\lambda_{x,i})},\ i = 1,2,3,\ |z| = 1 \right\} = \gamma \tag{3}$$

see Figure 1.

It is possible to construct truncated versions of the limiting 3×3-block-Toeplitz Jacobi operators in such a way that they commute and satisfy the algebraic equation defining the teardrop curve. The truncated operators take the form

$$\widetilde{X} := \begin{pmatrix} A_0^x & B_0^x & & & & & & \\ (B_0^x)^\intercal & A_1^x & B_1^x & & & & & \\ & (B_1^x)^\intercal & A^x & B^x & & & & \\ & & (B^x)^\intercal & \ddots & \ddots & & & \\ & & & \ddots & \ddots & B^x & & \\ & & & & (B^x)^\intercal & A^x & (b_1^x)^\intercal & \\ & & & & & b_1^x & a_1^x & (b_0^x)^\intercal \\ & & & & & & b_0^x & a_0^x \end{pmatrix}, \tag{4}$$

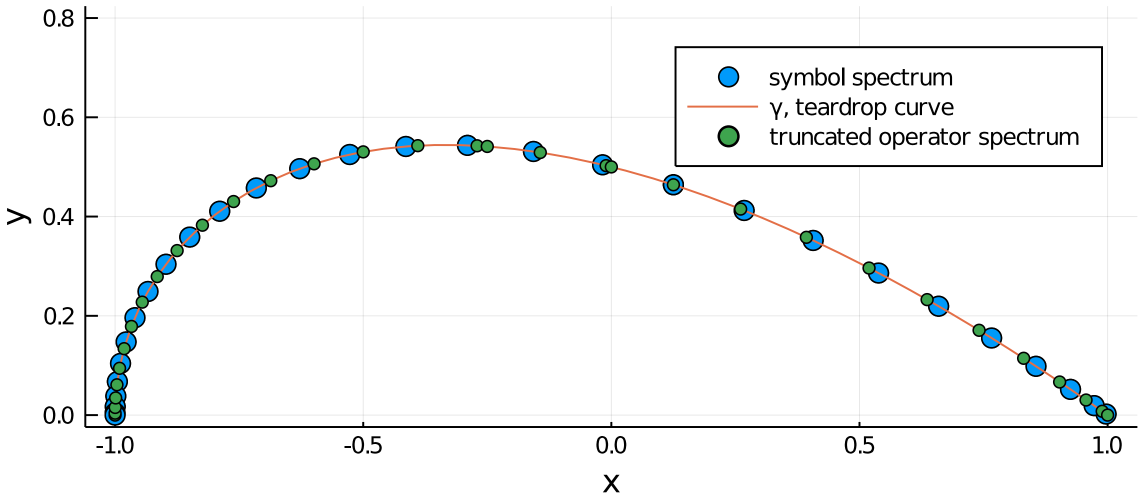

Figure 1: In blue, the joint spectrum of $X(z)$ and $Y(z)$, i.e., a plot of $(\lambda_{x,i}, \lambda_{y,i})$ defined in (3), sampled at 20 equally spaced points on the complex unit circle. In green, the joint spectrum of 33×33 versions of the truncated operators $\widetilde{X}$ and $\widetilde{Y}$ defined in (4)–(6).

where A_0^x, a_0^x are 1×1 matrices; B_0^x, b_0^x are 1×2; A_1^x, a_1^x are symmetric 2×2 matrices and B_1^x, b_1^x are 2×3 and A^x, B^x are the 3×3 matrices defined above. The truncated operator $\widetilde{Y}$ is defined similarly.

The entries of the block matrices in the top-left and bottom-right corners (A_0^x, a_0^x, A_0^y, a_0^y, etc.) are determined by requiring that

$$\widetilde{X}\widetilde{Y} = \widetilde{Y}\widetilde{X}, \qquad \widetilde{Y}^2 = \phi(\widetilde{X}) = \frac{1}{4}\left(\mathrm{I} - \widetilde{X}\right)^2 \left(\mathrm{I} + \widetilde{X}\right), \tag{5}$$

and that their joint spectrum lie on the support of the OPs. That is, we require

$$\widetilde{X} = Q\Lambda_x Q^\intercal, \qquad \widetilde{Y} = Q\Lambda_y Q^\intercal, \qquad \Lambda_y = \sqrt{\phi(\Lambda_x)}, \tag{6}$$

where Q is an orthogonal matrix.

For $\alpha = \beta = -1/2$, we have found a 4-parameter family of truncated operators that satisfy (5) and (6): A^x, B^x, A^y, B^y are given in (1) and (2);

$$A_0^x = \begin{pmatrix} x_1 \end{pmatrix}, \qquad B_0^x = \begin{pmatrix} 0 & 0 \end{pmatrix}, \qquad A_1^x = \begin{pmatrix} x_2 & 0 \\ 0 & -\frac{3}{8} \end{pmatrix}, \qquad B_1^x = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 \\ 4 & -1 & 0 \end{pmatrix},$$

where $x_1, x_2 \in [-1, 1]$;

$$A_0^y = \begin{pmatrix} y_1 \end{pmatrix}, \quad B_0^y = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad A_1^y = \begin{pmatrix} y_2 & 0 \\ 0 & 18v \end{pmatrix}, \quad B_1^y = v\begin{pmatrix} 0 & 0 & 0 \\ 2 & 6 & 1 \end{pmatrix}, \quad v = \frac{\sqrt{2}}{64},$$

where $y_i = \sqrt{\phi(x_i)}$, $i = 1, 2$;

$$a_0^x = \begin{pmatrix} x_3 \end{pmatrix}, \qquad b_0^x = \begin{pmatrix} 0 & 0 \end{pmatrix}, \qquad a_1^x = \begin{pmatrix} -\frac{3}{8} & 0 \\ 0 & x_4 \end{pmatrix}, \qquad b_1^x = -\frac{1}{8}\begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

where $x_3, x_4 \in [-1, 1]$ and

$$a_0^y = \begin{pmatrix} y_3 \end{pmatrix}, \quad b_0^y = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad a_1^y = \begin{pmatrix} 18v & 0 \\ 0 & y_4 \end{pmatrix}, \quad b_1^y = v\begin{pmatrix} -1 & 6 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \quad v = \frac{\sqrt{2}}{64},$$

where $y_i = \sqrt{\phi(x_i)}$, $i = 3, 4$. Figure 1 shows the joint spectrum of $\widetilde{X}$ and $\widetilde{Y}$ for the choices $x_1 = 1, x_2 = -1/4, x_3 = 0, x_4 = -1$.