Keplerian Orbits in Cartesian and Celestial Coordinates

NEW EARTH Lab – HAA

Jensen Lawrence

— Last Updated: February 9, 2022 —

Contents

1 Cartesian Coordinates **1**
1.1 Position 1
1.2 Velocity 3
1.3 Acceleration 5

2 Celestial Coordinates **7**
2.1 Choice of Reference Frame 7
2.2 Position 7
2.3 Velocity 8
2.4 Acceleration 8
2.5 Validity of the Small Angle Approximation 9

A Elliptical Geometry **12**

B The Keplerian Two-Body Problem **13**

C Kepler's Laws of Planetary Motion **16**

1 Cartesian Coordinates

1.1 Position

Consider a less massive object, like a planet (hereafter the "secondary") orbiting a more massive object, like a star (hereafter the "primary"). The orbit of the secondary is an ellipse described by the equation

$$r = \frac{a(1-e^2)}{1+e\cos(\nu)}. \tag{1.1}$$

Here, a is the semi-major axis of the orbital ellipse, e is the eccentricity of the orbital ellipse, and ν is the true anomaly of the secondary, which describes its angular position on the orbital ellipse. For a complete derivation of this result, see Appendix B and Eq. (B.25).

Since Cartesian coordinates are related to polar coordinates by $(x, y) = (r\cos(\nu), r\sin(\nu))$, then in the frame of the secondary, we have that

$$\mathbf{r} = \begin{bmatrix} r\cos(\nu) \\ r\sin(\nu) \\ 0 \end{bmatrix}. \tag{1.2}$$

However, the frame of the secondary and the frame of an external observer are unlikely to be coincident. Instead, the observer is often rotated with respect to the secondary. To translate from the frame of the secondary to the frame of the observer, we introduce the following rotation angles:

- Argument of pericentre (ω): describes how the orbital ellipse is rotated in the orbital plane relative to a reference direction selected by the observer.
- Inclination (i): describes how the orbital plane is tilted relate to a reference plane selected by the observer.
- Longitude of ascending node (Ω): describes how the orbital ellipse is rotated in the reference plane relative to a reference direction selected by the observer.

Figure 1 provides a visual demonstration of how these angles affect orbital orientation.

To transform between the orbital and external frames, we must apply the appropriate rotations by ω, i, and Ω. We begin by applying a rotation on $\mathbf{r}$ due to ω. This constitutes a rotation about the z-axis of the orbital plane, meaning we multiply $\mathbf{r}$ by $R_z(\omega)$, where R_z is the z-axis rotation matrix. Doing so, we obtain

$$\begin{aligned} \mathbf{r}' &= R_z(\omega)\mathbf{r}, \\ &= \begin{bmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos(\nu) \\ r\sin(\nu) \\ 0 \end{bmatrix}, \\ &= \begin{bmatrix} r\cos(\nu+\omega) \\ r\sin(\nu+\omega) \\ 0 \end{bmatrix}. \end{aligned} \tag{1.3}$$

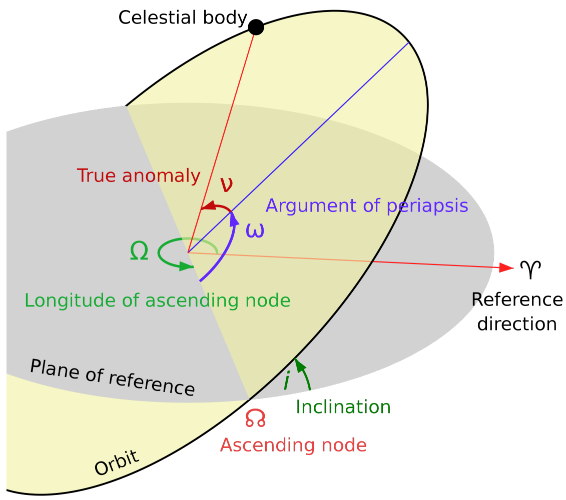


Figure 1: The effect of the angular orbital elements on the orientation of the orbital ellipse (source: Wikipedia)

Next, we apply a rotation on $\mathbf{r}'$ due to i. Care must be taken for this rotation, since the choice of reference direction and the sign of the rotation angle affects subsequent calculations. We choose to define the reference direction to align with the x-axis of the observer's coordinate system. As well, we choose to rotate by $+i$ as opposed to $-i$. This causes the z-component of the secondary's velocity to increase as it moves away from the observer, matching the corresponding increase in the redshift of the secondary's spectrum.

Thus, we rotate $\mathbf{r}'$ about the x-axis of the reference plane by i, meaning we multiply $\mathbf{r}'$ by $R_x(i)$, where R_x is the x-axis rotation matrix. This yields

$$\begin{aligned}\mathbf{r}'' &= R_x(i)\mathbf{r}, \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & -\sin(i) \\ 0 & \sin(i) & \cos(i) \end{bmatrix} \begin{bmatrix} r\cos(\nu+\omega) \\ r\sin(\nu+\omega) \\ 0 \end{bmatrix}, \\ &= \begin{bmatrix} r\cos(\nu+\omega) \\ r\sin(\nu+\omega)\cos(i) \\ r\sin(\nu+\omega)\sin(i) \end{bmatrix}. \end{aligned} \tag{1.4}$$

Finally, to obtain the observed position vector, we apply a rotation on $\mathbf{r}''$ due to Ω. This constitutes a rotation about the z-axis of the reference plane, meaning we multiply $\mathbf{r}$ by $R_z(\Omega)$. This gives us

$$\begin{aligned}\mathbf{r}_{\text{obs}} &= R_z(\Omega)\mathbf{r}'', \\ &= \begin{bmatrix} \cos(\Omega) & -\sin(\Omega) & 0 \\ \sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos(\nu+\omega) \\ r\sin(\nu+\omega)\cos(i) \\ r\sin(\nu+\omega)\sin(i) \end{bmatrix}, \\ &= \begin{bmatrix} r\cos(\nu+\omega)\cos(\Omega) - r\sin(\nu+\omega)\cos(i)\sin(\Omega) \\ r\cos(\nu+\omega)\sin(\Omega) + r\sin(\nu+\omega)\cos(i)\cos(\Omega) \\ r\sin(\nu+\omega)\sin(i) \end{bmatrix}. \end{aligned} \tag{1.5}$$

Summary: the position of the secondary as measured by the observer is

$$x_{\rm obs} = r\left[\cos(\nu+\omega)\cos(\Omega) - \sin(\nu+\omega)\cos(i)\sin(\Omega)\right], \tag{1.6}$$
$$y_{\rm obs} = r\left[\cos(\nu+\omega)\sin(\Omega) + \sin(\nu+\omega)\cos(i)\cos(\Omega)\right], \tag{1.7}$$
$$z_{\rm obs} = r\sin(\nu+\omega)\sin(i), \tag{1.8}$$

where r is given by Eq. (1.1).

1.2 Velocity

To determine the observed velocity components, we must compute $\dot{x}_{\rm obs}$, $\dot{y}_{\rm obs}$, and $\dot{z}_{\rm obs}$. This procedure can be simplified by calculating some preliminary results. To begin, we compute $r\dot{\nu}$. By Kepler's second law (combining Eqs. (C.1) and (C.2)), we have that

$$\frac{r^2\dot{\nu}}{2} = \frac{\pi a^2\sqrt{1-e^2}}{T}, \tag{1.9}$$

where a is the semi-major axis, e is the eccentricity, and T is the orbital period of the secondary. Rearranging this expression for $r\dot{\nu}$ and employing Eq. (1.1), we obtain

$$r\dot{\nu} = \frac{2\pi a}{T}\frac{1+e\cos(\nu)}{\sqrt{1-e^2}}. \tag{1.10}$$

Next, we calculate $\dot{r}$. Differentiating Eq. (1.1) yields

$$\dot{r} = \frac{r\dot{\nu}e\sin(\nu)}{1+e\cos(\nu)}. \tag{1.11}$$

Combining this with Eq. (1.10) produces

$$\dot{r} = \frac{2\pi a}{T}\frac{e\sin(\nu)}{\sqrt{1-e^2}}. \tag{1.12}$$

Using $\dot{r}$ and $r\dot{\nu}$, we can now differentiate $r\sin(\nu+\omega)$ and $r\cos(\nu+\omega)$. For $r\sin(\nu+\omega)$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[r\sin(\nu+\omega)] &= \dot{r}\sin(\nu+\omega) + r\dot{\nu}\cos(\nu+\omega),\\ &= \frac{2\pi a}{T}\frac{1}{\sqrt{1-e^2}}\left(\cos(\nu+\omega) + e\cos(\omega)\right). \end{aligned}\tag{1.13}$$

Next, for $r\cos(\nu+\omega)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[r\cos(\nu+\omega)] &= \dot{r}\cos(\nu+\omega) - r\dot{\nu}\sin(\nu+\omega),\\ &= -\frac{2\pi a}{T}\frac{1}{\sqrt{1-e^2}}\left(\sin(\nu+\omega) + e\sin(\omega)\right). \end{aligned}\tag{1.14}$$

Finally, for convenience, we define

$$J \equiv \frac{2\pi a}{T}\frac{1}{\sqrt{1-e^2}}, \tag{1.15}$$

allowing us to write

$$\frac{\mathrm{d}}{\mathrm{d}t}[r\sin(\nu+\omega)] = J\left(\cos(\nu+\omega)+e\cos(\omega)\right), \tag{1.16}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[r\cos(\nu+\omega)] = -J\left(\sin(\nu+\omega)+e\sin(\omega)\right). \tag{1.17}$$

Using these results, we can easily determine $\dot{x}_{\rm obs}$, $\dot{y}_{\rm obs}$, and $\dot{z}_{\rm obs}$. First, for $\dot{x}_{\rm obs}$, we get

$$\begin{aligned}\dot{x}_{\rm obs} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[r\left[\cos(\nu+\omega)\cos(\Omega)-\sin(\nu+\omega)\cos(i)\sin(\Omega)\right]\right],\\ &= \cos(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\left[r\cos(\nu+\omega)\right]-\cos(i)\sin(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\left[r\sin(\nu+\omega)\right],\\ &= -J\left[\cos(i)\sin(\Omega)\left(\cos(\nu+\omega)+e\cos(\omega)\right)+\cos(\Omega)\left(\sin(\nu+\omega)+e\sin(\omega)\right)\right]. \end{aligned} \tag{1.18}$$

Next, for $\dot{y}_{\rm obs}$, we get

$$\begin{aligned}\dot{y}_{\rm obs} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[r\left[\cos(\nu+\omega)\sin(\Omega)+\sin(\nu+\omega)\cos(i)\cos(\Omega)\right]\right],\\ &= \sin(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}[r\cos(\nu+\omega)]+\cos(i)\cos(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}[r\sin(\nu+\omega)],\\ &= J\left[\cos(i)\cos(\Omega)\left(\cos(\nu+\omega)+e\cos(\omega)\right)-\sin(\Omega)\left(\sin(\nu+\omega)+e\sin(\omega)\right)\right]. \end{aligned} \tag{1.19}$$

Finally, for $\dot{z}_{\rm obs}$, we get

$$\begin{aligned}\dot{z}_{\rm obs} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[r\sin(\nu+\omega)\sin(i)\right],\\ &= J\sin(i)\left(\cos(\nu+\omega)+e\cos(\omega)\right). \end{aligned} \tag{1.20}$$

To match radial velocity literature, we define the radial velocity semiamplitude K as

$$K \equiv J\sin(i) = \frac{2\pi a}{T}\frac{\sin(i)}{\sqrt{1-e^2}}, \tag{1.21}$$

allowing us to write $\dot{z}_{\rm obs}$ as

$$\dot{z}_{\rm obs} = K(\cos(\nu+\omega)+e\cos(\omega)). \tag{1.22}$$

Summary: the velocity of the secondary as measured by the observer is

$$\dot{x}_{\rm obs} = -J\left[\cos(i)\sin(\Omega)\left(\cos(\nu+\omega)+e\cos(\omega)\right)+\cos(\Omega)\left(\sin(\nu+\omega)+e\sin(\omega)\right)\right], \tag{1.23}$$
$$\dot{y}_{\rm obs} = J\left[\cos(i)\cos(\Omega)\left(\cos(\nu+\omega)+e\cos(\omega)\right)-\sin(\Omega)\left(\sin(\nu+\omega)+e\sin(\omega)\right)\right], \tag{1.24}$$
$$\dot{z}_{\rm obs} = K(\cos(\nu+\omega)+e\cos(\omega)), \tag{1.25}$$

where J and K are given by Eqs. (1.15) and (1.21), respectively.

1.3 Acceleration

To determine the observed acceleration components, we must compute $\ddot{x}_{\rm obs}$, $\ddot{y}_{\rm obs}$, and $\ddot{z}_{\rm obs}$. Like with the velocity, this procedure can be simplified by calculating some preliminary results. To begin, we compute $\dot{\nu}$. Combining Eqs. (1.1) and (1.10), we obtain

$$\dot{\nu} = \frac{2\pi}{T}\frac{(1+e\cos(\nu))^2}{(1-e^2)^{3/2}}. \tag{1.26}$$

Next, we differentiate $J\sin(\nu+\omega)$ and $J\cos(\nu+\omega)$. For $J\sin(\nu+\omega)$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[J\sin(\nu+\omega)\right] &= J\dot{\nu}\cos(\nu+\omega),\\ &= \frac{2\pi}{T}\frac{J}{(1-e^2)^{3/2}}(1+e\cos(\nu))^2\cos(\nu+\omega).\end{aligned} \tag{1.27}$$

Next, for $J\cos(\nu+\omega)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[J\cos(\nu+\omega)\right] &= -J\dot{\nu}\sin(\nu+\omega),\\ &= -\frac{2\pi}{T}\frac{J}{(1-e^2)^{3/2}}(1+e\cos(\nu))^2\sin(\nu+\omega).\end{aligned} \tag{1.28}$$

Finally, for convenience, we define

$$A \equiv \frac{2\pi}{T}\frac{J}{(1-e^2)^{3/2}} = \frac{4\pi^2 a}{T^2}\frac{1}{(1-e^2)^2},, \tag{1.29}$$

allowing us to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[J\sin(\nu+\omega)\right] = A(1+e\cos(\nu))^2\cos(\nu+\omega), \tag{1.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[J\cos(\nu+\omega)\right] = -A(1+e\cos(\nu))^2\sin(\nu+\omega). \tag{1.31}$$

Using these results, we can easily determine $\ddot{x}_{\rm obs}$, $\ddot{y}_{\rm obs}$, and $\ddot{z}_{\rm obs}$. First, for $\ddot{x}_{\rm obs}$, we get

$$\begin{aligned}\ddot{x}_{\rm obs} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[-J\left[\cos(i)\sin(\Omega)\left(\cos(\nu+\omega)+e\cos(\omega)\right)+\cos(\Omega)\left(\sin(\nu+\omega)+e\sin(\omega)\right)\right]\right],\\ &= -\cos(i)\sin(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\left[J\cos(\nu+\omega)\right]-\cos(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\left[J\sin(\nu+\omega)\right],\\ &= A(1+e\cos(\nu))^2\left[\cos(i)\sin(\Omega)\sin(\nu+\omega)-\cos(\Omega)\cos(\nu+\omega)\right].\end{aligned} \tag{1.32}$$

Next, for $\ddot{y}_{\rm obs}$, we get

$$\begin{aligned}\ddot{y}_{\rm obs} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[J\left[\cos(i)\cos(\Omega)\left(\cos(\nu+\omega)+e\cos(\omega)\right)-\sin(\Omega)\left(\sin(\nu+\omega)+e\sin(\omega)\right)\right]\right],\\ &= \cos(i)\cos(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\left[J\cos(\nu+\omega)\right]-\sin(\Omega)\frac{\mathrm{d}}{\mathrm{d}t}\left[J\sin(\nu+\omega)\right],\\ &= -A(1+e\cos(\nu))^2\left[\cos(i)\cos(\Omega)\sin(\nu+\omega)+\sin(\Omega)\cos(\nu+\omega)\right].\end{aligned} \tag{1.33}$$

Finally, recalling that $K = J\sin(i)$, for $\ddot{z}_{\rm obs}$ we get

$$\begin{aligned}
\ddot{z}_{\rm obs} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[K(\cos(\nu+\omega)+e\cos(\omega))\right], \\
&= \sin(i)\frac{\mathrm{d}}{\mathrm{d}t}\left[J\cos(\nu+\omega)\right], \\
&= -A\sin(i)(1+e\cos(\nu))^2\sin(\nu+\omega). \qquad (1.34)
\end{aligned}$$

We define the radial acceleration semiamplitude B as

$$B \equiv A\sin(i) = \frac{2\pi}{T}\frac{J\sin(i)}{(1-e^2)^{3/2}} = \frac{2\pi}{T}\frac{K}{(1-e^2)^{3/2}} = \frac{4\pi^2 a}{T^2}\frac{\sin(i)}{(1-e^2)^2}, \qquad (1.35)$$

allowing us to write $\ddot{z}_{\rm obs}$ as

$$\ddot{z}_{\rm obs} = -B(1+e\cos(\nu))^2\sin(\nu+\omega). \qquad (1.36)$$

Summary: the acceleration of the secondary as measured by the observer is

$$\begin{aligned}
\ddot{x}_{\rm obs} &= A(1+e\cos(\nu))^2\left[\cos(i)\sin(\Omega)\sin(\nu+\omega)-\cos(\Omega)\cos(\nu+\omega)\right], \qquad (1.37)\\
\ddot{y}_{\rm obs} &= -A(1+e\cos(\nu))^2\left[\cos(i)\cos(\Omega)\sin(\nu+\omega)+\sin(\Omega)\cos(\nu+\omega)\right], \qquad (1.38)\\
\ddot{z}_{\rm obs} &= -B(1+e\cos(\nu))^2\sin(\nu+\omega), \qquad (1.39)
\end{aligned}$$

where A and B are given by Eqs. (1.29) and (1.35), respectively.

2 Celestial Coordinates

2.1 Choice of Reference Frame

For this work, we choose the reference plane to be the plane orthogonal to the observer's line of site. As well, we choose the reference direction to be the celestial north pole. This means that, in reference to the work done in the previous part, the positive x-axis points in the direction of positive declination (upwards on the sky), while the positive y-axis points in the direction of positive right ascension (left on the sky). This choice of coordinates is important to note, since it differs from the right/up orientation of the x/y axes seen elsewhere.

2.2 Position

Although knowledge of x_{obs} and y_{obs} are useful theoretically, they are not as useful in practice, since distances are challenging to measure in astronomy. Angular separation, however, is much easier to measure, and therefore much more useful for observational applications.

Consider two objects a distance d away from an observer. If the observer measures that they are separated by a distance r, then the angular separation of the two objects measured by the observer is

$$\Delta\theta = \arctan\left(\frac{r}{d}\right). \tag{2.1}$$

Note that for $x \ll 1$, $\arctan(x) \approx x$. Thus, if $r \ll d$, we can employ the small angle approximation, which states that

$$\Delta\theta \approx \frac{r}{d}. \tag{2.2}$$

We want to use these results to determine the angular separation of a secondary from its primary. Specifically, we want to know the right ascension offset $\Delta\alpha$ and the declination offset $\Delta\delta$ of the secondary from its primary. Combining Eq. (2.1) with the coordinate conventions from 2.1, we see that these offsets are given by

$$\Delta\alpha = \arctan\left(\frac{y_{\text{obs}}}{d}\right) \quad \text{and} \quad \Delta\delta = \arctan\left(\frac{x_{\text{obs}}}{d}\right). \tag{2.3}$$

If $x_{\text{obs}}, y_{\text{obs}} \ll d$, then we can apply the small angle approximation, giving us

$$\Delta\alpha \approx \frac{y_{\text{obs}}}{d} \quad \text{and} \quad \Delta\delta \approx \frac{x_{\text{obs}}}{d}. \tag{2.4}$$

Summary: the right ascension and declination offsets of the secondary from the primary as measured by the observer are

$$\Delta\alpha = \arctan\left(\frac{y_{\text{obs}}}{d}\right) \approx \frac{y_{\text{obs}}}{d} \quad \text{if } y_{\text{obs}} \ll d, \tag{2.5}$$

$$\Delta\delta = \arctan\left(\frac{x_{\text{obs}}}{d}\right) \approx \frac{x_{\text{obs}}}{d} \quad \text{if } x_{\text{obs}} \ll d. \tag{2.6}$$

where d is the distance to the primary, and x_{obs} and y_{obs} are given by Eqs. (1.6) and (1.7), respectively.

2.3 Velocity

To determine the right ascension and declination offset velocities of the secondary, we must first derive a general expression for $\Delta\dot{\theta}$. Differentiating Eq. (2.1), we get

$$\begin{aligned} \Delta\dot{\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\arctan\left(\frac{r}{d}\right)\right], \\ &= \frac{\dot{r}}{d}\frac{1}{1+(r/d)^2}. \end{aligned} \tag{2.7}$$

To determine $\Delta\dot{\theta}$ in the regime of the small angle approximation, differentiating Eq. (2.2) to obtain $\Delta\dot{\theta} \approx \dot{r}/d$ is the most straightforward approach. However, for confirmation, we also apply the small angle approximation directly to Eq. (2.7). Note that if $r \ll d$, then $(r/d)^2 \ll 1$, meaning $1/(1+(r/d)^2) \approx 1$. Thus, we are left with

$$\Delta\dot{\theta} \approx \frac{\dot{r}}{d}, \tag{2.8}$$

confirming the expected result. Since $r = y_{\mathrm{obs}}$ for $\Delta\alpha$ and $r = x_{\mathrm{obs}}$ for $\Delta\delta$, then the right ascension and declination offset velocities are given by

$$\Delta\dot{\alpha} = \frac{\dot{y}_{\mathrm{obs}}}{d}\frac{1}{1+(y_{\mathrm{obs}}/d)^2} \quad \text{and} \quad \Delta\dot{\delta} = \frac{\dot{x}_{\mathrm{obs}}}{d}\frac{1}{1+(x_{\mathrm{obs}}/d)^2}. \tag{2.9}$$

If $x_{\mathrm{obs}}, y_{\mathrm{obs}} \ll d$, then we can apply the small angle approximation, giving us

$$\Delta\dot{\alpha} \approx \frac{\dot{y}_{\mathrm{obs}}}{d} \quad \text{and} \quad \Delta\dot{\delta} \approx \frac{\dot{x}_{\mathrm{obs}}}{d}. \tag{2.10}$$

Summary: the right ascension and declination offset velocities of the secondary as measured by the observer are

$$\Delta\dot{\alpha} = \frac{\dot{y}_{\mathrm{obs}}}{d}\frac{1}{1+(y_{\mathrm{obs}}/d)^2} \approx \frac{\dot{y}_{\mathrm{obs}}}{d} \quad \text{if } y_{\mathrm{obs}} \ll d, \tag{2.11}$$

$$\Delta\dot{\delta} = \frac{\dot{x}_{\mathrm{obs}}}{d}\frac{1}{1+(x_{\mathrm{obs}}/d)^2} \approx \frac{\dot{x}_{\mathrm{obs}}}{d} \quad \text{if } x_{\mathrm{obs}} \ll d, \tag{2.12}$$

where d is the distance to the primary, x_{obs} and y_{obs} are given by Eqs. (1.6) and (1.7), and $\dot{x}_{\mathrm{obs}}$ and $\dot{y}_{\mathrm{obs}}$ are given by Eqs. (1.23) and (1.24), respectively.

2.4 Acceleration

To determine the right ascension and declination offset accelerations of the secondary, we must first derive a general expression for $\Delta\ddot{\theta}$. Differentiating Eq. (2.7), we get

$$\begin{aligned} \Delta\ddot{\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\dot{r}}{d}\frac{1}{1+(r/d)^2}\right], \\ &= \frac{d[\ddot{r}(d^2+r^2) - 2r\dot{r}^2]}{(d^2+r^2)^2}. \end{aligned} \tag{2.13}$$

To determine $\Delta\ddot{\theta}$ in the regime of the small angle approximation, differetiating Eq. (2.2) twice to obtain $\Delta\ddot{\theta} \approx \ddot{r}/d$ is the most straightforward approach. However, for confirmation, we also apply the small angle approximation directly to Eq. (2.13). We can rewrite this equation as

$$\Delta\ddot{\theta} = \frac{(\ddot{r}/d) + [(\ddot{r} - 2\dot{r})/r](r/d)^3}{1 + 2(r/d)^2 + (r/d)^4}. \tag{2.14}$$

Note that if $r \ll d$, then $(r/d)^2$, $(r/d)^3$, and $(r/d)^4$ are each much less than unity, meaning they can be neglected. This leaves us with

$$\Delta\ddot{\theta} \approx \frac{\ddot{r}}{d}, \tag{2.15}$$

confirming the expected result. Since $r = y_{\text{obs}}$ for $\Delta\alpha$ and $r = x_{\text{obs}}$ for $\Delta\delta$, then the right ascension and declination offset accelerations are given by

$$\Delta\ddot{\alpha} = \frac{d[\ddot{y}_{\text{obs}}(d^2 + y_{\text{obs}}^2) - 2y_{\text{obs}}\dot{y}_{\text{obs}}^2]}{(d^2 + y_{\text{obs}}^2)^2} \quad \text{and} \quad \Delta\ddot{\delta} = \frac{d[\ddot{x}_{\text{obs}}(d^2 + x_{\text{obs}}^2) - 2x_{\text{obs}}\dot{x}_{\text{obs}}^2]}{(d^2 + x_{\text{obs}}^2)^2}. \tag{2.16}$$

If $x_{\text{obs}}, y_{\text{obs}} \ll d$, then we can apply the small angle approximation, giving us

$$\Delta\ddot{\alpha} \approx \frac{\ddot{y}_{\text{obs}}}{d} \quad \text{and} \quad \Delta\ddot{\delta} \approx \frac{\ddot{x}_{\text{obs}}}{d}. \tag{2.17}$$

Summary: the right ascension and declination offset accelerations of the secondary as measured by the observer are

$$\Delta\ddot{\alpha} = \frac{d[\ddot{y}_{\text{obs}}(d^2 + y_{\text{obs}}^2) - 2y_{\text{obs}}\dot{y}_{\text{obs}}^2]}{(d^2 + y_{\text{obs}}^2)^2} \approx \frac{\ddot{y}_{\text{obs}}}{d} \quad \text{if } y_{\text{obs}} \ll d, \tag{2.18}$$

$$\Delta\ddot{\delta} = \frac{d[\ddot{x}_{\text{obs}}(d^2 + x_{\text{obs}}^2) - 2x_{\text{obs}}\dot{x}_{\text{obs}}^2]}{(d^2 + x_{\text{obs}}^2)^2} \approx \frac{\ddot{x}_{\text{obs}}}{d} \quad \text{if } x_{\text{obs}} \ll d, \tag{2.19}$$

where d is the distance to the primary, x_{obs} and y_{obs} are given by Eqs. (1.6) and (1.7), $\dot{x}_{\text{obs}}$ and $\dot{y}_{\text{obs}}$ are given by Eqs. (1.23) and (1.24), and $\ddot{x}_{\text{obs}}$ and $\ddot{y}_{\text{obs}}$ are given by Eqs. (1.37) and (1.38) respectively.

2.5 Validity of the Small Angle Approximation

The validity of the small angle approximation, especially for the cross-derivative terms in the angular acceleration equation, may seem questionable. It is therefore prudent to compare exact and approximate results across a wide range of parameter values to ensure the small angle approximation is applicable.

For exoplanetary systems, the semi-major axis to distance ratio a/d is at least 10^5, if not 10^6 or more for most distant systems. We therefore choose to test the small angle approximation on the interval $a/d \in [10^5, 10^8]$. For thoroughness, we test over all possible eccentricities, using the interval $e \in [0, 0.995]$. For each combination of a/d and e, we compute the values of $\Delta\theta$, $\Delta\dot{\theta}$, and $\Delta\ddot{\theta}$ over one orbit using the exact and small angle formulas. We then compute the maximum absolute error between the exact and small angle results to quanity the performance of the small angle approximation. The results of this exercise are shown in the following graphs.

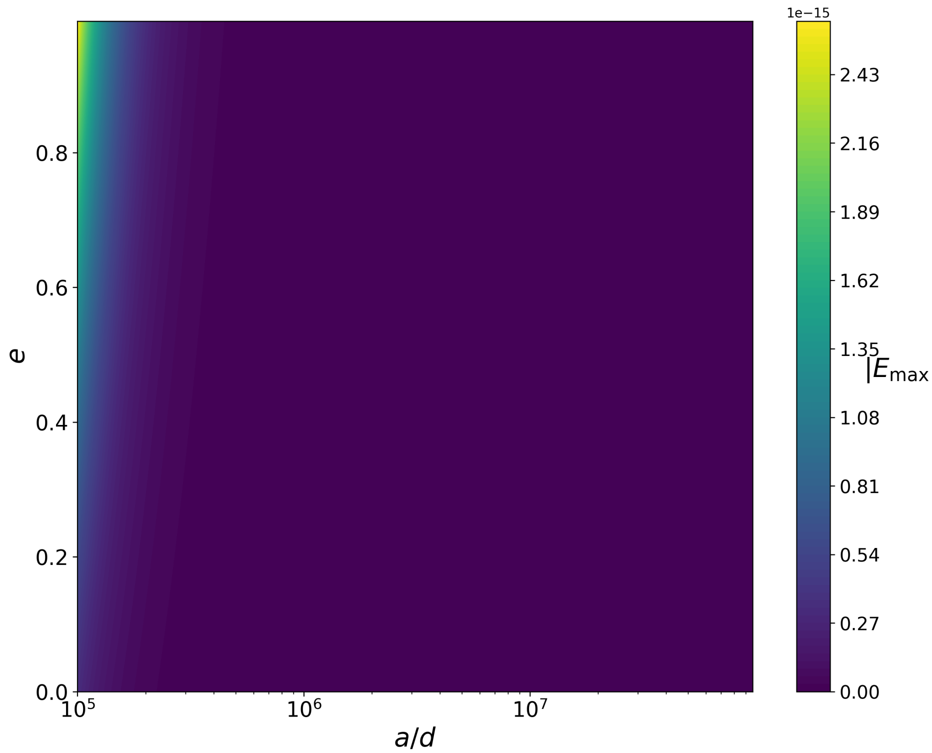


Figure 2: Maximum absolute error in $\Delta\theta$

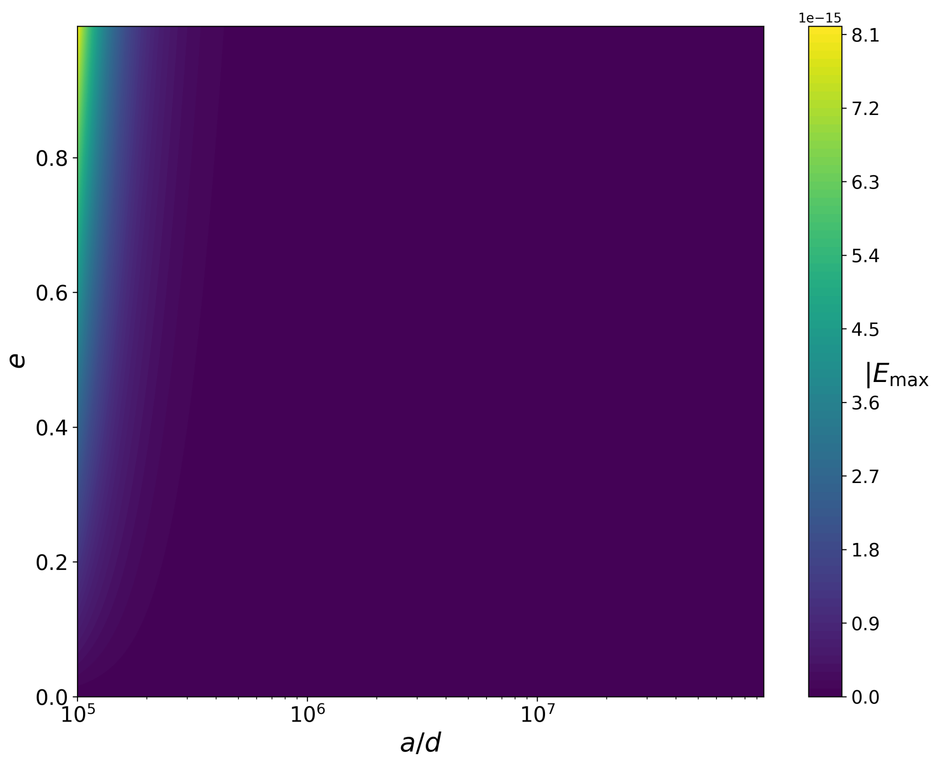


Figure 3: Maximum absolute error in $\Delta\dot{\theta}$

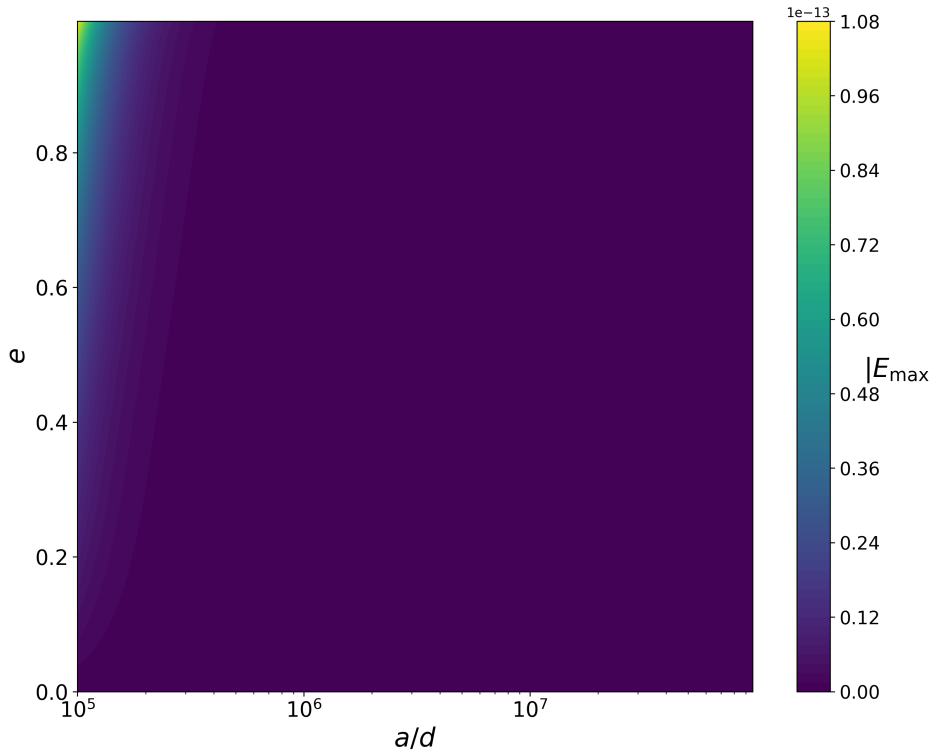


Figure 4: Maximum absolute error in $\Delta\ddot{\theta}$

As we can see in these figures, the error of the small angle approximation is on the order of 10^{-14} to 10^{-15}. This error is so small it approaches machine precision, making the small angle approximation essentially indistinguishable from the exact formulas. Thus, the small angle approximation is valid when applied to exoplanetary systems.

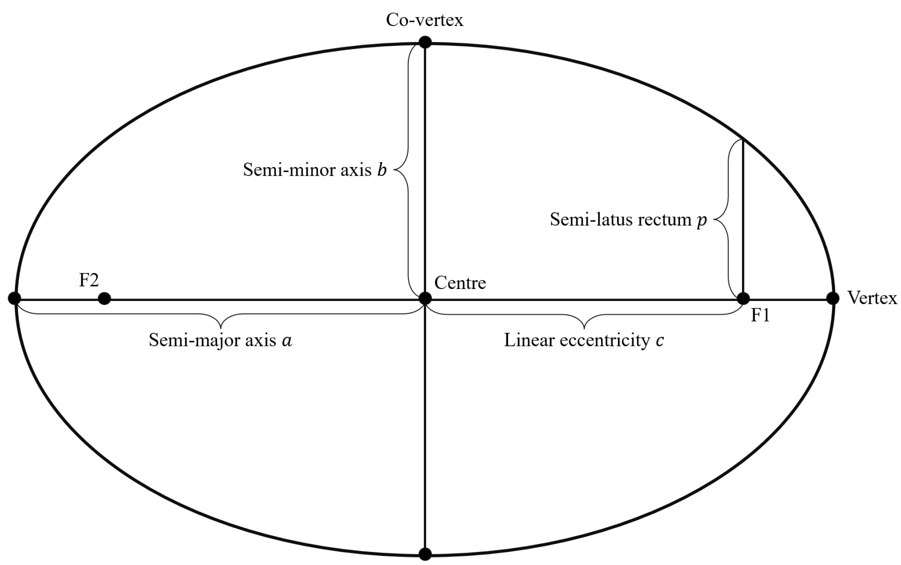
A Elliptical Geometry

As will be seen in the following section, all bounded, closed Keplerian orbits are ellipses. This makes understanding elliptical geometry necessary for understanding Keplerian orbits.

Consider an ellipse of width $2a$ and height $2b$ centred at (x_0, y_0) (without loss of generality, we assume that $a \geq b$). The equation of such an ellipse in Cartesian coordinates is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1. \tag{A.1}$$

This ellipse appears as follows:



The semi-major and semi-minor axes, a and b, are the distances from the centre to the vertex and co-vertex, respectively. The two foci, F1 and F2, are located at $(\pm c + x_0, y_0)$, where c is given by

$$c = \sqrt{a^2 - b^2}. \tag{A.2}$$

c is called the linear eccentricity, and is a measure of the ellipse's elongation. A more common measure of the ellipse's elongation is the eccentricity e, which is given by

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}. \tag{A.3}$$

Note that if given the semi-major axis and eccentricity, the distance to each focus can then be determined according to the equation

$$c = ea. \tag{A.4}$$

Finally, the semi-latus rectum p is the vertical distance from either focus to the ellipse, and has the value

$$p = a(1 - e^2). \tag{A.5}$$

The quantities e, p, and c are used in the characterization of Keplerian orbits.

B The Keplerian Two-Body Problem

Consider two particles with masses m_1 and m_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ interacting only through Newtonian gravity. The individual kinetic energies of these particles are

$$T_1 = \frac{1}{2}m_1\|\dot{\mathbf{r}}_1\|^2 \quad \text{and} \quad T_2 = \frac{1}{2}m_2\|\dot{\mathbf{r}}_2\|^2. \tag{B.1}$$

As well, the gravitational potential energy between the particles is

$$V = -\frac{Gm_1m_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|}, \tag{B.2}$$

where G is the gravitational constant. Therefore, the Lagrangian of this two-body system is

$$\mathcal{L} = \frac{1}{2}m_1\|\mathbf{r}_1\|^2 + \frac{1}{2}m_2\|\mathbf{r}_2\|^2 + \frac{Gm_1m_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|}. \tag{B.3}$$

We now define the relative position $\mathbf{r}$ and centre of mass position $\mathbf{R}$ as

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2, \tag{B.4}$$

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \tag{B.5}$$

where $M \equiv M_1 + M_2$ is the total mass. By inspection, we see that we can now write $\mathbf{r}_1$ and $\mathbf{r}_2$ as

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}. \tag{B.6}$$

Substituting these relations into the Lagrangian (B.3), we obtain

$$\mathcal{L} = \frac{1}{2}M\|\dot{\mathbf{R}}\|^2 + \frac{1}{2}\mu\|\dot{\mathbf{r}}\|^2 + \frac{Gm_1m_2}{\|\mathbf{r}\|}, \tag{B.7}$$

where $\mu \equiv (m_1m_2)/M$ is the reduced mass. Notice that there are no $\mathbf{r}, \mathbf{R}$ cross-terms, meaning we can separate the total Lagrangian $\mathcal{L}$ into the centre of mass Lagrangian $\mathcal{L}_{\text{com}}$ and the relative Lagrangian $\mathcal{L}_{\text{rel}}$. These are given by

$$\mathcal{L}_{\text{com}} = \frac{1}{2}M\|\dot{\mathbf{R}}\|^2, \tag{B.8}$$

$$\mathcal{L}_{\text{rel}} = \frac{1}{2}\mu\|\dot{\mathbf{r}}\|^2 + \frac{Gm_1m_2}{\|\mathbf{r}\|}. \tag{B.9}$$

Applying the Euler-Lagrange equations to the centre of mass Lagrangian (B.8), we see that

$$\frac{\partial \mathcal{L}_{\text{com}}}{\partial \mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}_{\text{com}}}{\partial \dot{\mathbf{R}}} \implies M\ddot{\mathbf{R}} = \mathbf{0}. \tag{B.10}$$

In other words, the centre of mass of the two-body system does not accelerate.

We now want to apply the Euler-Lagrange equations to the relative Lagrangian (B.9). Before proceeding, we can reduce the dimensionality of the problem. Note that the angular momentum of the two-body system is

$$\mathbf{L} = \mathbf{r}_1 \times (m_1\dot{\mathbf{r}}_1) + \mathbf{r}_2 \times (m_2\dot{\mathbf{r}}_2) = \mu(\mathbf{r} \times \dot{\mathbf{r}}). \tag{B.11}$$

Since angular momentum is conserved, $\mathbf{L}$ must be constant in time. Since μ is constant, then $\mathbf{r} \times \dot{\mathbf{r}}$ must also be constant. This implies that the motion of the system is restricted to a plane, thus reducing the problem from three dimensions to two dimensions. We can therefore express $\mathbf{r}$ in terms of the polar coordinates (r, ν), where r is the separation between the two particles and ν is the angle formed by $\mathbf{r}$ relative to some reference. This allows us to write $\mathcal{L}_{\text{rel}}$ as

$$\mathcal{L}_{\text{rel}} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\nu}^2) + \frac{Gm_1m_2}{r}. \tag{B.12}$$

Evaluating the Euler-Lagrange equation for ν yields

$$\frac{\partial \mathcal{L}_{\text{rel}}}{\partial \nu} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}_{\text{rel}}}{\partial \dot{\nu}} \implies \mu r^2 \dot{\nu} = \text{const.} \tag{B.13}$$

Note that $L \equiv \|\mathbf{L}\| = \mu r^2 \dot{\nu}$ is the (constant) system angular momentum, so

$$\dot{\nu} = \frac{L}{\mu r^2}. \tag{B.14}$$

Evaluating the Euler-Lagrange equation for r and using the expression for $\dot{\nu}$, we obtain

$$\frac{\partial \mathcal{L}_{\text{rel}}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}_{\text{rel}}}{\partial \dot{r}} \implies \mu \ddot{r} = \frac{L^2}{\mu r^3} - \frac{Gm_1m_2}{r^2}. \tag{B.15}$$

To simplify the evaluation of this differential equation, we let $\rho = 1/r$, and then solve the resulting equation for $\rho(\nu)$, as opposed to $\rho(t)$. Under these changes, we get the equation

$$-\frac{L^2\rho^2}{\mu}\frac{\mathrm{d}^2\rho}{\mathrm{d}\nu^2} = \frac{L^2}{\mu}\rho^3 - Gm_1m_2\rho^2 \implies \frac{\mathrm{d}^2\rho}{\mathrm{d}\nu^2} + \rho = \frac{Gm_1m_2\mu}{L^2}. \tag{B.16}$$

The solution to this equation is

$$\rho(\nu) = A\cos(\nu - B) + \frac{Gm_1m_2\mu}{L^2}, \tag{B.17}$$

where A and B are constants of integration. Since B amounts to a phase angle, we can rotate our coordinates and set $B = 0$ without loss of generality. Furthermore, if we define $\xi = L^2/(Gm_1m_2\mu)$ and $\chi = A\xi$, we can write the solution as

$$\rho(\nu) = \xi^{-1}(1 + \chi\cos(\nu)). \tag{B.18}$$

Finally, since $r = 1/\rho$ by definition, we see that

$$r = \frac{\xi}{1 + \chi\cos(\nu)}. \tag{B.19}$$

We have thus determined an equation for the separation r between the two particles as a function of the angle ν. ξ is determined by the physical properties of the system, while χ is determined by the initial conditions of the system.

We will now limit ourselves to the case of closed, bounded orbits, which is the case of interest for planets and many other objects that orbit stars. By inspection, we see that Eq. (B.19) is closed and bounded when $0 \leq \chi < 1$, so we restrict the value of χ accordingly.

Note that we can transform between polar coordinates and Cartesian coordinates using

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \cos(\nu) = \frac{x}{\sqrt{x^2 + y^2}}. \tag{B.20}$$

Substituting these conversions into Eq. (B.19) and rearranging (while assuming $0 \le \chi < 1$), we obtain the expression

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{B.21}$$

where

$$a = \frac{\xi}{1-\chi^2}, \quad b = \frac{\xi}{\sqrt{1-\chi^2}}, \quad \text{and} \quad d = \frac{\xi\chi}{1-\chi^2} = \chi a. \tag{B.22}$$

Comparing Eqs. (B.21) and (A.1), we see that Eq. (B.21) is the equation for an ellipse with semi-major axis a and semi-minor axis b, centred at $(-d, 0)$. Thus, bounded and closed solutions to the Keplerian two-body problem are ellipses.

Using the formula for eccentricity from Eq. (A.3) and the values of a and b from Eq. (B.22), we get that the eccentricity of the orbital ellipse is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \chi. \tag{B.23}$$

Since $\chi = e$, then from Eq. (A.5), the semi-latus rectum of the orbital ellipse is

$$p = a(1-\chi^2) = \xi, \tag{B.24}$$

which means that $\xi = a(1-e^2)$. Therefore, closed, bounded Keplerian orbits are described by the equation

$$r = \frac{a(1-e^2)}{1+e\cos(\nu)}, \tag{B.25}$$

Furthermore,

$$d = \chi a = ea, \tag{B.26}$$

and by Eq. (A.4) we know that $c = ea$. Thus, the centre of the orbital ellipse is at $(-c, 0)$. This implies that the F1 focus of the ellipse is at $(0, 0)$, and so the centre of mass of the two-particle system is located at the focus of the orbital ellipse.

C Kepler's Laws of Planetary Motion

From Eq. (B.25), we can determine Kepler's laws of planetary motion. These laws are:

(1) Every planetary orbit is an ellipse with the star-planet centre of mass at one of the two foci. Note that most planets are significantly less massive than their stars, meaning the location of the centre of mass is approximately the same as the centre of the star.

(2) A line joining a planet to its star will sweep out equal area in equal time. Specifically, the change in swept area is related to the angular velocity of the planet by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{r^2}{2}\frac{\mathrm{d}\nu}{\mathrm{d}t}, \tag{C.1}$$

from which we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\pi ab}{T} = \frac{\pi a^2\sqrt{1-e^2}}{T}, \tag{C.2}$$

where $b = a\sqrt{1-e^2}$ is the semi-minor axis of the orbital ellipse (see Eq. (A.3)) and T is the orbital period of the planet.

(3) The square of a planet's orbital period is proportional to the cube of its semi-major axis. Specifically, we have

$$T^2 = \frac{4\pi^2}{G(M_\star + M_\mathrm{p})}a^3, \tag{C.3}$$

where $M_\star$ is the mass of the star and M_p is the mass of the planet. Once again, most planets are significantly less massive than their stars, giving us the approximate relation

$$T^2 \approx \frac{4\pi^2}{GM_\star}a^3. \tag{C.4}$$