Applications of Differential Geometry to Physics

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Contents

1 Introduction to Differential Forms

Lect. 1

This course will be somewhat different from the course given by Prof Gary Gibbons in previous years. We will plan to cover applications of differential geometry in general relativity, quantum field theory, and string theory.

1.1 Vectors, Tensors and p-forms

Assume we have some kind of d-dimensional manifold, possibly representing spacetime, with a set of co-ordinates x^a, $a = 1, \ldots, d$.
In general relativity, typically one thinks of a **vector** as being represented by u^a. But u^a is really the components of a vector in some particular basis. We need to think about basis-independent expressions.
In d dimensions, there is always a set of d basis vectors

$$E_1, \ldots, E_d, \quad \text{collectively } E_a. \tag{1.1}$$

A vector is then

$$u = \sum_a u^a E_a, \tag{1.2}$$

where u^a are the components of u in the basis $\{E_a\}$.
A **one-form** ω is an object which is dual to a vector, i.e. given a vector u and a one-form ω there is a bracket operation $\langle \omega, u \rangle$ giving a real number.
This bracket is linear: If $u = \alpha v + \beta w$ for arbitrary vectors v, w and real numbers α, β,

$$\langle \omega, \alpha v + \beta w \rangle = \alpha \langle \omega, v \rangle + \beta \langle \omega, w \rangle. \tag{1.3}$$

We can write a one-form as

$$\omega = \sum_a \omega_a E^a, \tag{1.4}$$

where ω_a are numbers and E^a are one-forms. Then the bracket can be defined as

$$\langle E^a, E_b \rangle = \delta^a{}_b, \tag{1.5}$$

such that the basis of one-forms are dual to the basis of vectors.
The bracket is also linear in ω: If $\omega = \alpha\eta + \beta\lambda$ for one-forms η, λ and real numbers α, β,

$$\langle \alpha\eta + \beta\lambda, u \rangle = \alpha \langle \eta, u \rangle + \beta \langle \lambda, u \rangle. \tag{1.6}$$

Consider now

$$\langle \omega, u \rangle = \sum_{a,b} \langle \omega_a E^a, u^b E_b \rangle = \sum_{a,b} \omega_a u^b \langle E^a, E_b \rangle = \sum_a \omega_a u^a. \tag{1.7}$$

The bracket corresponds to the usual scalar multiplication.
The next thing is to define the derivative of a function $f(x)$, denoted by df - this is a one-form. It should have the property

$$\langle df, X \rangle = X\, f. \tag{1.8}$$

We can pick a set of one-forms and a basis of vectors to make it explicit. In a **co-ordinate** basis, these basis vectors are $\frac{\partial}{\partial x^i}$ and the one-forms are dx^i. These are dual,

$$\langle \frac{\partial}{\partial x^i}, dx^j \rangle = \delta^i{}_j, \tag{1.9}$$

which is consistent with the definition of df, since

$$\langle \frac{\partial}{\partial x^i}, dx^j \rangle = \frac{\partial}{\partial x^i} x^j. \tag{1.10}$$

This can also be done for an arbitrary vector $X = X^j \frac{\partial}{\partial x^j}$. From linearity,

$$\langle df, X \rangle = \langle df, X^j \frac{\partial}{\partial x^j} \rangle = X^j \frac{\partial}{\partial x^j} f. \tag{1.11}$$

This is the **directional derivative** of f in the direction X.
This, roughly speaking, is what one-forms are. There is a simple geometrical consequence; suppose that

$$\langle df, X \rangle = 0. \tag{1.12}$$

Then f is a constant in the direction of the vector X, which means that df is normal to surfaces of $f =$ constant.
We can put this into a bigger perspective: Functions f are often called 0-forms. Then df, the derivative of f, is a one-form. We have defined an operator d turning 0-forms into one-forms. In general, d will turn p-forms into $(p+1)$-forms. In terms of a co-ordinate basis,

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{1.13}$$

This is exactly as expected from the chain rule for a derivative.

A general **tensor** is of type (r, s); its components are $T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}$. We think of this as something which does not depend on a basis:

$$T = T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}\, E_{a_1} \otimes E_{a_2} \otimes \ldots \otimes E_{a_r} \otimes E^{b_1} \otimes \ldots \otimes E^{b_s}. \tag{1.14}$$

This is independent of the particular basis in question.
In general relativity, a tensor transforms in a particular way under a co-ordinate transformation. But this is really just a change of basis:

$$E_a \to E_{a'} = \chi_{a'}{}^a E_a, \tag{1.15}$$

where $\chi_{a'}{}^a$ represents a non-degenerate $d \times d$ matrix. Similarly, one could do a transformation on the basis one-forms

$$E^a \to E^{a'} = \Phi^{a'}{}_a E^a. \tag{1.16}$$

This could be a co-ordinate basis, but does not have to be. Looking at the bracket, we must have

$$\delta^{a'}{}_{b'} = \langle E^{a'}, E_{b'} \rangle = \langle \Phi^{a'}{}_a E^a, \chi_{b'}{}^b E_b \rangle = \Phi^{a'}{}_a \chi_{b'}{}^b \delta^a{}_b = \Phi^{a'}{}_a \chi_{b'}{}^a, \tag{1.17}$$

thus χ is the matrix inverse of Φ. Under a change of basis, the tensor T must be invariant, thus

$$\begin{aligned} T &= T^{a'_1 \ldots a'_r}{}_{b'_1 \ldots b'_s} E_{a'_1} \otimes E_{a'_2} \otimes \ldots \otimes E_{a'_r} \otimes E^{b'_1} \otimes \ldots \otimes E^{b'_s} \\ &= T^{a'_1 \ldots a'_r}{}_{b'_1 \ldots b'_s} \chi_{a'_1}{}^{a_1} \ldots \chi_{a'_r}{}^{a_r} \Phi^{b'_1}{}_{b_1} \ldots \Phi^{b'_s}{}_{b_s} E_{a_1} \otimes E_{a_2} \otimes \ldots \otimes E_{a_r} \otimes E^{b_1} \otimes \ldots \otimes E^{b_s} \\ &= T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} E_{a_1} \otimes E_{a_2} \otimes \ldots \otimes E_{a_r} \otimes E^{b_1} \otimes \ldots \otimes E^{b_s}, \end{aligned} \tag{1.18}$$

so the components of T transform as (expressing the old components in terms of the new)

$$T^{a'_1 \ldots a'_r}{}_{b'_1 \ldots b'_s} \chi_{a'_1}{}^{a_1} \ldots \chi_{a'_r}{}^{a_r} \Phi^{b'_1}{}_{b_1} \ldots \Phi^{b'_s}{}_{b_s} = T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s}, \tag{1.19}$$

exactly as expected from the co-ordinate formulation of general relativity.
A p-**form** is defined to be a tensor of type $(0,p)$ whose components are totally antisymmetric (in any basis):

$$T = T_{a_1 \ldots a_p} E^{a_1} \otimes \ldots \otimes E^{a_p} = T_{a_1 \ldots a_p} E^{[a_1} \otimes \ldots \otimes E^{a_p]} = \frac{1}{p!} T_{a_1 \ldots a_p} \left(E^{a_1} \wedge \ldots \wedge E^{a_p} \right), \tag{1.20}$$

where we define the wedge product

$$E^{a_1} \wedge \ldots \wedge E^{a_p} := \sum_{\sigma \in \mathfrak{S}_p} \pi(\sigma) E^{\sigma(a_1)} \otimes E^{\sigma(a_2)} \otimes \ldots \otimes E^{\sigma(a_p)} \tag{1.21}$$

and the sum is over all permutations σ of p elements with parity $\pi(\sigma)$ either $+1$ or -1, so there are $p!$ terms in the sum. $\wedge$ basically tells you to take the antisymmetric product:

$$\begin{aligned} E^a \wedge E^b &= E^a \otimes E^b - E^b \otimes E^a, \\ E^a \wedge E^b \wedge E^c &= E^a \otimes E^b \otimes E^c + E^b \otimes E^c \otimes E^a + E^c \otimes E^a \otimes E^b \\ &\quad - E^a \otimes E^c \otimes E^b - E^b \otimes E^a \otimes E^c - E^c \otimes E^b \otimes E^a, \end{aligned} \tag{1.22}$$

etc. $E^{a_1} \wedge \ldots \wedge E^{a_p}$ is antisymmetric under the interchange of any adjacent pair of indices. In d dimensions, the number of linearly independent such objects is

$$\frac{d(d-1)\ldots(d-p+1)}{p!} = \frac{d!}{p!(d-p)!} = \binom{d}{p}. \tag{1.23}$$

This means one must have $p \leq d$, because one will get nothing otherwise.

1.2 Operations on Forms

Lect. 2

The next thing is to look at a product of a p-form P and a q-form Q. A p-form P can in any basis be written as

$$P = \frac{1}{p!} P_{a_1 \ldots a_p} E^{a_1} \wedge E^{a_2} \wedge \ldots \wedge E^{a_p}, \tag{1.24}$$

similarly

$$Q = \frac{1}{q!} Q_{b_1 \ldots b_q} E^{b_1} \wedge E^{b_2} \wedge \ldots \wedge E^{b_q}. \tag{1.25}$$

We already have a rule for defining the product of one-forms. We define the **wedge product** of a p-form with a q-form to be

$$P \wedge Q = \frac{1}{(p+q)!} P_{a_1 \ldots a_p} Q_{b_1 \ldots b_q} E^{a_1} \wedge E^{a_2} \wedge \ldots \wedge E^{a_p} \wedge E^{b_1} \wedge E^{b_2} \wedge \ldots \wedge E^{b_q}. \tag{1.26}$$

You can think of this in a slightly different way. $P \wedge Q$ is really equivalent to a tensor of type $(0, p+q)$ that is antisymmetric on all its $p+q$ indices. If you wanted to know its components, you could write down a simple formula

$$P_{[a_1 \ldots a_p} Q_{b_1 \ldots b_q]}. \tag{1.27}$$

That, of course, means that if you stare at this product, consequently

$$P \wedge Q = (-)^{pq} Q \wedge P. \tag{1.28}$$

We have discovered that differential forms have a $\mathbb{Z}_2$**-grading**:

$$P \wedge Q = \begin{cases} Q \wedge P & \text{if either } p \text{ or } q \text{ is even,} \\ -Q \wedge P & \text{if } p \text{ and } q \text{ are odd} \end{cases}. \tag{1.29}$$

You can think of P or Q as odd objects if p, q are odd, and as even objects if p or q are even. (This is analogous to bosons which are described by even quantum fields, and fermions which are described by odd quantum fields in quantum field theory.)

To avoid possible ambiguities, we write out explicitly what is meant by $[\cdot]$, namely antisymmetrization with weight one:

$$X_{[a_1 \ldots a_p]} = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \pi(\sigma) X_{\sigma(a_1) \ldots \sigma(a_p)}, \tag{1.30}$$

so that

$$\begin{aligned} X_{[ab]} &= \frac{1}{2} \left(X_{ab} - X_{ba} \right), \\ X_{[abc]} &= \frac{1}{6} \left(X_{abc} + X_{bca} + X_{cab} - X_{acb} - X_{bac} - X_{cba} \right), \end{aligned} \tag{1.31}$$

etc. Similarly, $(\cdot)$ always means symmetrization with weight one.

The next thing is to define an **exterior derivative** d on p-forms. We look at a p-form in a co-ordinate basis:

$$P = \frac{1}{p!} \underbrace{P_{a_1 \ldots a_p}}_{\text{set of } 0-\text{forms}} \underbrace{dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p}}_{p-\text{form}}. \tag{1.32}$$

We already know what d does on 0-forms, and so we define

$$dP = \frac{1}{p!}\frac{\partial P_{a_1\ldots a_p}}{\partial x^b} dx^b \wedge dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_p}. \tag{1.33}$$

This is consistent with how d acts on a 0-form to give a one-form. There is an alternative convention where dx^b is put at the end which gives you unpleasant factors of $(-)^p$, and which we will not use. Because

$$dP = X = \frac{1}{(p+1)!} X_{[a_1\ldots a_{p+1}]} dx^{a_1} \wedge dx^{a_2} \wedge \ldots \wedge dx^{a_{p+1}}, \tag{1.34}$$

we can write the components of $X = dP$ in terms of the components of $\frac{\partial P}{\partial x}$:

$$X_{a_1\ldots a_{p+1}} = (-)^p (p+1) \partial_{[a_{p+1}} P_{a_1\ldots a_p]}, \tag{1.35}$$

where we write ∂_a for $\frac{\partial}{\partial x^a}$. It is impossible to suppress all factors of $(-)^p$; this one is a nuisance.
Properties of the operator d:

- d maps p-forms to $(p+1)$-forms. To see this, you have to prove that dP is a tensor. Do the calculation in a co-ordinate basis: Under a change of co-ordinates $x^a \to x'^{a'} = x'^{a'}(x^a)$, we define

$$A^{a'}{}_a = \frac{\partial x'^{a'}}{\partial x^a}, \quad A_{a'}{}^a = \frac{\partial x^a}{\partial x'^{a'}}. \tag{1.36}$$

Then if P is a p-form,

$$P_{a_1\ldots a_p} \to P_{a'_1\ldots a'_p} = A_{a'_1}{}^{a_1} A_{a'_2}{}^{a_2} \ldots A_{a'_p}{}^{a_p} P_{a_1\ldots a_p}. \tag{1.37}$$

The components of dP transform as

$$\begin{aligned}
\partial_{[b} P_{a_1\ldots a_p]} \to \partial_{[b'} P_{a'_1\ldots a'_p]} &= \partial_{[b'} \left(A_{a'_1}{}^{a_1} A_{a'_2}{}^{a_2} \ldots A_{a'_p]}{}^{a_p} P_{a_1\ldots a_p} \right) \\
&= \frac{\partial x^b}{\partial x'^{[b'|}} \partial_b \left(A_{|a'_1}{}^{a_1} A_{a'_2}{}^{a_2} \ldots A_{a'_p]}{}^{a_p} P_{a_1\ldots a_p} \right) \\
&= A_{b'}{}^b A_{a'_1}{}^{a_1} A_{a'_2}{}^{a_2} \ldots A_{a'_p}{}^{a_p} \partial_{[b} P_{a_1\ldots a_p]} \\
&\quad + A_{[b'|}{}^b \partial_b \left(A_{|a'_1}{}^{a_1} \right) A_{a'_2}{}^{a_2} \ldots A_{a'_p]}{}^{a_p} P_{a_1\ldots a_p} + \ldots,
\end{aligned} \tag{1.38}$$

with more similar terms. These all contain terms of the form

$$\frac{\partial x^b}{\partial x'^{b'}} \frac{\partial A_{a'_1}{}^{a_1}}{\partial x^b} = \frac{\partial x^b}{\partial x'^{b'}} \frac{\partial x^{a_1}}{\partial x'^{a'_1} \partial x^b} = \frac{\partial x^{a_1}}{\partial x'^{a'_1} \partial x'^{b'}}, \tag{1.39}$$

antisymmetrized over a'_1 and b'. Since partial derivatives commute, these terms all vanish. What you end up with is what you expect for a tensorial object:

$$\partial_{[b} P_{a_1\ldots a_p]} \to A_{b'}{}^b A_{a'_1}{}^{a_1} A_{a'_2}{}^{a_2} \ldots A_{a'_p}{}^{a_p} \partial_{[b} P_{a_1\ldots a_p]}. \tag{1.40}$$

Components of dP transform tensorially under a co-ordinate transformation.

- $d^2 = 0$. This is most easily seen by looking at the components of $d(dP)$.

$$\begin{aligned} \text{components of } P &\sim P_{[a_1 \dots a_p]} \\ \text{components of } dP &\sim \partial_{[b} P_{a_1 \dots a_p]} \\ \text{components of } d(dP) &\sim \partial_{[c} \partial_{[b} P_{a_1 \dots a_p]]} = \partial_{[c} \partial_b P_{a_1 \dots a_p]} = 0. \end{aligned} \tag{1.41}$$

  Remember there was a $\mathbb{Z}_2$-grading. dP is a $(p+1)$-form and so d changes the $\mathbb{Z}_2$-grading of the form.
  So morally, d had better be odd. Therefore $dd = -dd = 0$.

- The operator d is Leibnizian.

$$\begin{aligned} P &= P_{a_1 \dots a_p} dx^{a_1} \otimes \dots \otimes dx^{a_p} \\ dP &= \underbrace{dP_{a_1 \dots a_p}}_{\frac{\partial P_{a_1 \dots a_p}}{\partial x^b} dx^b} \wedge dx^{a_1} \otimes \dots \otimes dx^{a_p} + \dots, \end{aligned} \tag{1.42}$$

  where all remaining terms contain some ddx^{a_i} and will vanish.

- d acting on the product of a p-form with a q-form:
  The components of $P \wedge Q$ are

$$P_{[a_1 \dots a_p} Q_{b_1 \dots b_q]}. \tag{1.43}$$

  Then the components of $d(P \wedge Q)$ will be proportional to

$$\partial_{[b} (P_{a_1 \dots a_p}) Q_{b_1 \dots b_q]} + (P_{[a_1 \dots a_p}) \partial_b Q_{b_1 \dots b_q]}. \tag{1.44}$$

  Since $X_{[a_1 \dots a_p b b_1 \dots b_q]} = (-)^p X_{[b a_1 \dots a_p b_1 \dots b_q]}$, this shows that

$$d(P \wedge Q) = dP \wedge Q + (-)^p P \wedge dQ. \tag{1.45}$$

- All manipulations were in a co-ordinate basis but this is inessential. The action of d is independent of a choice of co-ordinates.

This all looks like messing about, but it is easy to apply these things to electromagnetism, Yang-Mills theory and general relativity. As of now, the word "metric" has not been mentioned. Forms, their products and their exterior derivatives are all concepts which are <u>independent</u> of the metric. We will need an object called the **alternating tensor**: This is an object $\varepsilon^{a_1 \dots a_d}$ which is antisymmetric under the interchange of any adjacent pair of indices. It has components

$$\varepsilon^{a_1 \dots a_d} = \frac{1}{\sqrt{|g|}} \begin{cases} +1 & (a_1 \dots a_d) \text{ is an even permutation of } (1, \dots, d) \\ -1 & (a_1 \dots a_d) \text{ is an odd permutation of } (1, \dots, d) \\ 0 & \text{otherwise.} \end{cases} \tag{1.46}$$

Here $g = \det g_{ab}$ for a metric g_{ab}. These form the components of a rank d tensor (proof provided later). One can also form

$$\varepsilon_{a_1 \dots a_d} = g_{a_1 b_1} g_{a_2 b_2} \cdots g_{a_d b_d} \varepsilon^{b_1 \dots b_d}; \tag{1.47}$$

this has components

$$\varepsilon_{a_1 \dots a_d} = (-)^t \sqrt{|g|} \begin{cases} +1 & (a_1 \dots a_d) \text{ is an even permutation of } (1, \dots, d) \\ -1 & (a_1 \dots a_d) \text{ is an odd permutation of } (1, \dots, d) \\ 0 & \text{otherwise.} \end{cases} \tag{1.48}$$

Here t is the number of timelike directions, which may be different depending on the type of geometry one is studying.
Pure mathematicians study almost exclusively **Riemannian geometry** - this is based on the axiom that if the distance $\int ds$, as defined by the metric

$$ds^2 = g_{ab} dx^a dx^b, \tag{1.49}$$

between two points is zero, then they are the same point.
This means that the metric g is positive definite, with only positive eigenvalues. The signature is $(+^d)$. This type of geometry is known in the physics literature, quite confusingly, as "Euclidean". It corresponds to $t = 0$.
We contrast this with what happens in general relativity, where one studies **pseudo-Riemannian geometry**. Here g is not positive-definite and $ds = 0$ defines how light rays propagate. Typically, we have signature $(+^{d-1}, -)$, and $t = 1$. (The term **spacetime** means a manifold with such a metric in the following.)

There is also **Kleinian geometry**, which is encountered in twistor theory ($t = 3$) or in the F-theory approach to string theory ($t = 2$). Here one has a general signature $(+^p, -^t)$. One must remember this when doing calculations with forms.

Lect. 3

Now we prove that ε is indeed a tensor. That means that under a co-ordinate transformation

$$x^a \to x'^{a'} = x'^{a'}(x^a), \quad A^{a'}{}_a = \frac{\partial x'^{a'}}{\partial x^a}, \tag{1.50}$$

it must transform as

$$\begin{aligned} \varepsilon^{a'b'c'\dots} &= A^{a'}{}_a A^{b'}{}_b A^{c'}{}_c \dots \varepsilon^{abc\dots} \\ \frac{1}{\sqrt{|g'|}} \eta^{a'b'c'\dots} &= A^{a'}{}_a A^{b'}{}_b A^{c'}{}_c \dots \frac{1}{\sqrt{|g|}} \eta^{abc\dots}, \end{aligned} \tag{1.51}$$

where we defined the **alternating symbol** (not a tensor!)

$$\eta^{a_1 \dots a_d} = \begin{cases} +1 & (a_1 \dots a_d) \text{ is an even permutation of } (1, \dots, d) \\ -1 & (a_1 \dots a_d) \text{ is an odd permutation of } (1, \dots, d) \\ 0 & \text{otherwise.} \end{cases} \tag{1.52}$$

Because of the symmetry, there is really only one equation that has to be satisfied. We multiply the equation by $\eta^{a'b'c'\dots}$ and sum over all indices:

$$\sum_{a'b'c'\dots} \frac{1}{\sqrt{|g'|}} \eta^{a'b'c'\dots} \eta^{a'b'c'\dots} = \sum_{a'b'c'\dots} A^{a'}{}_a A^{b'}{}_b A^{c'}{}_c \dots \frac{1}{\sqrt{|g|}} \eta^{a'b'c'\dots} \eta^{abc\dots} \tag{1.53}$$

The sum on the left-hand side gives $d!$, on the right-hand side we have

$$\sum_{a'b'c'\ldots} A^{a'}{}_a A^{b'}{}_b A^{c'}{}_c \ldots \eta^{a'b'c'\ldots} = \eta^{abc\ldots} \det A, \tag{1.54}$$

and the remaining summation over $a, b, c, \ldots$ gives

$$(d!)\frac{1}{\sqrt{|g'|}} = (d!)\frac{1}{\sqrt{|g|}} \det A. \tag{1.55}$$

For that to be true we must have that under a co-ordinate transformation

$$|g'| = |g|(\det A)^{-2}. \tag{1.56}$$

Since the metric is a tensor, it transforms as

$$g'_{a'b'} = \frac{\partial x^a}{\partial x'^{a'}}\frac{\partial x^b}{\partial x'^{b'}} g_{ab} = A_{a'}{}^a A_{b'}{}^b g_{ab}, \tag{1.57}$$

where $A_{a'}{}^a$ is the inverse of A. Take the determinant of this equation to get

$$\det g' = \det(A^{-1}A^{-1}g) = (\det A)^{-2} \det g. \tag{1.58}$$

Putting this together leads to the conclusion that ε really is tensorial. ε has the following useful properties:

$$\varepsilon^{abcd\ldots}\varepsilon_{abcd\ldots} = (-)^t d!; \tag{1.59}$$

from that, you can derive other contractions, such as

$$\begin{aligned} \varepsilon^{abc\ldots de}\varepsilon_{abc\ldots df} &= (-)^t (d-1)! \delta^e{}_f, \\ \varepsilon^{ab\ldots c}\varepsilon_{pq\ldots r} &= (-)^t d! \delta^{[a}{}_{[p}\delta^b{}_q \ldots \delta^{c]}{}_{r]}. \end{aligned} \tag{1.60}$$

Now we want to construct the **dual** of a differential form. We start off with a p-form P; its dual is going to be $*P$, a $(d-p)$-form. We define this in terms of its components, in any basis: If

$$P = \frac{1}{p!} P_{a_1\ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p}, \tag{1.61}$$

we define

$$*P = \frac{1}{(d-p)!}(*P)_{a_1\ldots a_{d-p}} dx^{a_1} \wedge \ldots \wedge dx^{a_{d-p}}, \tag{1.62}$$

where

$$(*P)_{a_1\ldots a_{d-p}} = \frac{1}{p!}\varepsilon_{a_1\ldots a_{d-p}}{}^{b_1\ldots b_p} P_{b_1\ldots b_p}. \tag{1.63}$$

Note that we contract the last p indices, this is conventional. We can construct the double dual of P, and find that its components are

$$\begin{aligned} (**P)_{c_1\ldots c_p} &= \frac{1}{p!(d-p)!}\varepsilon_{c_1\ldots c_p}{}^{a_1\ldots a_{d-p}}\varepsilon_{a_1\ldots a_{d-p}}{}^{b_1\ldots b_p} P_{b_1\ldots b_p} \\ &= \frac{(-)^{p(d-p)}}{p!(d-p)!}\varepsilon_{a_1\ldots a_{d-p}c_1\ldots c_p}\varepsilon^{a_1\ldots a_{d-p}b_1\ldots b_p} P_{b_1\ldots b_p} \\ &= \frac{(-)^{p(d-p)}}{p!(d-p)!}(-)^t (d-p)! \delta^{[b_1}{}_{[c_1}\delta^{b_2}{}_{c_2}\ldots\delta^{b_p]}{}_{c_p]} P_{b_1\ldots b_p} \\ &= (-)^{p(d-p)}(-)^t P_{b_1\ldots b_p} \end{aligned} \tag{1.64}$$

and hence we obtain

$$**P = (-)^{p(d-p)+t}P. \tag{1.65}$$

This means that if t is even, then

$$**P = \begin{cases} -P & \text{if } d \text{ even and } p \text{ odd} \\ P & \text{otherwise;} \end{cases} \tag{1.66}$$

for odd t it is the other way around.

1.3 Electromagnetism and Yang-Mills Theory

Now we will find a use for forms. The simplest use for forms is Maxwell's equations, where now $d = 4, t = 1$. These are

$$\nabla_{[a}F_{bc]} = 0 \quad \Leftrightarrow \quad \partial_{[a}F_{bc]} = 0; \qquad \nabla_a F^{ab} = -j^b. \tag{1.67}$$

We can rewrite this in terms of forms, this will make life easier:

$$dF = 0, \quad *d*F = -j. \tag{1.68}$$

We do this explicitly. F is an antisymmetric tensor, the field strength. We can therefore construct a two-form

$$F = \frac{1}{2}F_{ab}dx^a \wedge dx^b; \tag{1.69}$$

then (remember $d^2 \equiv 0$)

$$\begin{aligned} dF &= d\left(\frac{1}{2}F_{ab}dx^a \wedge dx^b\right) \\ &= \frac{1}{2}dF_{ab} \wedge dx^a \wedge dx^b \\ &= \frac{1}{2}\frac{\partial F_{ab}}{\partial x^c}dx^c \wedge dx^a \wedge dx^b \\ &= \frac{1}{2}\left(\partial_{[c}F_{ab]}dx^c \wedge dx^a \wedge dx^b\right) = 0 \end{aligned} \tag{1.70}$$

reproduces the first set of equations. We need to define a current one-form for the other half of Maxwell's equations:

$$j = j_a dx^a. \tag{1.71}$$

Now work out $*d*F$:

$$*F = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}dx^a \wedge dx^b\right). \tag{1.72}$$

We will "cheat" by using Riemann normal co-ordinates. In these co-ordinates,

$$g \sim \eta, \quad \Gamma \sim 0, \quad \partial\Gamma \neq 0. \tag{1.73}$$

All quantities are tensorial, so the results will hold in general. In these co-ordinates $d\varepsilon = 0$; then

$$\begin{aligned} d*F &= \frac{1}{4}d\left(\varepsilon_{ab}{}^{cd}F_{cd}dx^a \wedge dx^b\right) \\ &= \frac{1}{4}\varepsilon_{[ab}{}^{cd}\partial_{e]}F_{cd}dx^e \wedge dx^a \wedge dx^b. \end{aligned} \tag{1.74}$$

This is a three-form with components

$$(d*F)_{eab} = \frac{3}{2}\varepsilon_{[ab}{}^{cd}\partial_{e]}F_{cd}. \tag{1.75}$$

Then the components of $*d*F$ are

$$\begin{aligned}(*d*F)_p &= \frac{1}{6}\varepsilon_p{}^{eab}\frac{3}{2}\varepsilon_{[ab}{}^{cd}\partial_{e]}F_{cd} \\ &= \frac{1}{4}\varepsilon^{ab}{}_p{}^e\varepsilon_{ab}{}^{cd}\partial_e F_{cd} \\ &= -\delta^{[c|}{}_p g^{e|d]}\partial_e F_{cd} \\ &= -\delta^c{}_p g^{ed}\partial_e F_{cd} = -\partial^d F_{pd} = \partial^d F_{dp}.\end{aligned} \tag{1.76}$$

On the example sheet, you can do this with combinatorial factors and using Christoffel symbols for a general metric.
The simplest example is a current flowing through a wire in the z-direction. In cylindrical coordinates, the metric is

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\theta^2 + dz^2, \tag{1.77}$$

so that $\det g = -\rho^2$. The current density only has a z component

$$j_z = I\delta^{(2)}(\rho), \quad j = I\delta^{(2)}(\rho)dz. \tag{1.78}$$

We need to figure out F. The only component of the electromagnetic field is $B_\theta(\rho)$. That is

$$F_{\rho z} = -F_{z\rho} = -B_\theta. \tag{1.79}$$

The two-form will be

$$F = \frac{1}{2}F_{ab}dx^a \wedge dx^b = \frac{1}{2}\left(F_{\rho z}d\rho \wedge dz - F_{z\rho}dz \wedge d\rho\right) = F_{\rho z}d\rho \wedge dz = -B_\theta d\rho \wedge dz. \tag{1.80}$$

Then automatically

$$dF = -dB_\theta \wedge d\rho \wedge dz = -\frac{\partial B_\theta}{\partial \rho}d\rho \wedge d\rho \wedge dz = 0. \tag{1.81}$$

$*F$ has components

$$(*F)_{ab} = \frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd} = \frac{1}{2}\varepsilon^{pqcd}F_{cd}g_{ap}g_{bq}, \tag{1.82}$$

the only component will be

$$(*F)_{t\theta} = -(*F)_{\theta t} = \frac{1}{2}\left(\varepsilon^{t\theta\rho z}F_{\rho z}g_{tt}g_{\theta\theta} + \varepsilon^{t\theta z\rho}F_{z\rho}g_{tt}g_{\theta\theta}\right) = -\frac{1}{\rho}F_{\rho z}(-1)\rho^2 = -\rho B_\theta. \tag{1.83}$$

Then $*F = -\rho B_\theta dt \wedge d\theta$ and

$$d*F = -d(\rho B_\theta)dt \wedge d\theta = -\frac{\partial(\rho B_\theta)}{\partial\rho}d\rho \wedge dt \wedge d\theta. \tag{1.84}$$

Lect. 4

The only component of $*d*F$ is

$$*d*F = \varepsilon_{z\rho t\theta}(d*F)^{\rho t\theta}dz = \rho g^{\rho\rho}g^{tt}g^{\theta\theta}\left(-\frac{\partial}{\partial\rho}(\rho B_\theta)\right) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\theta)dz \tag{1.85}$$

Maxwell's equations are

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\theta) = -I\delta^{(2)}(\rho) \tag{1.86}$$

which gives the obvious result. So that is how you do electromagnetism.

Next consider a generalisation of electromagnetism, developed by Yang and Mills in 1954, and earlier (1952) by R. Shaw of the University of Hull.
Normally, a one-form A is

$$A = A_a dx^a \tag{1.87}$$

with functions A_a. But there is no requirement that A_a should be real-valued functions; they could be elements of a Lie algebra.
Take some Lie group G. There will be a set of generators in the adjoint representation $\{T_\alpha\}$.
The Cartan metric on the Lie algebra of G is

$$\eta_{\alpha\beta} = -2\mathrm{Tr}(T_\alpha T_\beta). \tag{1.88}$$

There will be compact and non-compact directions in general. Compact directions will be represented by anti-Hermitian generators for which $\eta_{\alpha\beta} = +1$; non-compact directions will be represented by Hermitian generators for which $\eta_{\alpha\beta} = -1$. This might fit in more with the mathematics than the physics literature, that is simply too bad. For physical Yang-Mills theories, G is compact as required to make a unitary quantum field theory.
The metric can then be used to raise or lower indices in the Lie algebra.
The group can be specified by the commutation relations

$$[T_\alpha, T_\beta] = c_{\alpha\beta}{}^{\gamma} T_\gamma, \tag{1.89}$$

where $c_{\alpha\beta}{}^{\gamma}$ are structure constants of the Lie algebra. Then we define

$$A_a = A_a^\alpha T_\alpha, \tag{1.90}$$

where A_a^α are components of the gauge field in question, and A is a Lie algebra valued one-form. This generalises the vector potential of electromagnetism. We need to find the analogue of the field strength. In electromagnetism, the field strength is invariant under gauge transformations. This requirement is too strong in Yang-Mills theory. We define

$$F = dA + gA \wedge A, \tag{1.91}$$

which is now a Lie algebra valued two-form, and g is a coupling constant that one introduces in particle physics. In the mathematics literature, one sets $g = 1$. In the field theory world, this is written out in terms of components:

$$\begin{aligned}
\frac{1}{2}F_{ab}^\alpha dx^a \wedge dx^b T_\alpha &= d(A_a^\alpha T_\alpha) \wedge dx^a + gA_a^\alpha A_b^\beta dx^a \wedge dx^b T_\alpha T_\beta \\
&= (dA_a^\alpha \wedge dx^a)T_\alpha + \frac{1}{2}gA_a^\alpha A_b^\beta dx^a \wedge dx^b [T_\alpha, T_\beta] \\
&= \partial_b A_a^\alpha dx^b \wedge dx^a T_\alpha + \frac{1}{2}gA_a^\alpha A_b^\beta dx^a \wedge dx^b [T_\alpha, T_\beta] \\
&= \frac{1}{2}\left(\partial_a A_b^\alpha - \partial_b A_a^\alpha\right) dx^a \wedge dx^b T_\alpha + \frac{1}{2}gA_a^\alpha A_b^\beta c_{\alpha\beta}{}^{\gamma}\left(dx^a \wedge dx^b\right) T_\gamma,
\end{aligned} \tag{1.92}$$

so

$$F^\alpha_{ab} = \partial_a A^\alpha_b - \partial_b A^\alpha_a + g A^\beta_a A^\gamma_b c_{\beta\gamma}{}^\alpha. \tag{1.93}$$

It is often simpler to do abstract calculations using forms.
In electromagnetism, since $F = dA$, one has automatically $dF = 0$. In Yang-Mills theory,

$$D_A F = 0, \tag{1.94}$$

where D_A is a gauge covariant derivative defined by

$$D_A F = dF + g[A, F], \tag{1.95}$$

where the commutator of a p-form P and a q-form Q is defined by

$$[P, Q] = \begin{cases} P \wedge Q - Q \wedge P & \text{if either or both of } P \text{ and } Q \text{ are even,} \\ P \wedge Q + Q \wedge P & \text{if } P \text{ and } Q \text{ are both odd.} \end{cases} \tag{1.96}$$

Substitute this in $F = dA + gA \wedge A$ to discover that $D_A F = 0$ (**Bianchi identity**):

$$\begin{aligned} D_A F &= d(dA + gA \wedge A) + g(A \wedge (dA + gA \wedge A) - (dA + gA \wedge A) \wedge A) \\ &= g dA \wedge A - gA \wedge dA + gA \wedge dA + g^2 A \wedge A \wedge A - g dA \wedge A - g^2 A \wedge A \wedge A \\ &= 0. \end{aligned} \tag{1.97}$$

Let us generalise gauge transformations: In electromagnetism, these are

$$A \to A + d\epsilon, \quad F \to F. \tag{1.98}$$

Here

$$A \to A + D_A \epsilon = A + d\epsilon + g[A, \epsilon]. \tag{1.99}$$

Then the infinitesimal change in F is

$$\begin{aligned} \delta F &= d\delta A + g\delta A \wedge A + gA \wedge \delta A \\ &= d(d\epsilon + gA\epsilon - g\epsilon A) + g(d\epsilon + gA\epsilon - g\epsilon A) \wedge A + gA \wedge (d\epsilon + gA\epsilon - g\epsilon A) \\ &= g dA\epsilon - gA \wedge d\epsilon - g d\epsilon \wedge A - g\epsilon dA + g d\epsilon \wedge A + g^2 A\epsilon \wedge A - g^2 \epsilon A \wedge A + gA \wedge d\epsilon \\ &\quad + g^2 A \wedge A\epsilon - g^2 A \wedge \epsilon A \\ &= g(dA + gA \wedge A)\epsilon - g\epsilon(dA + gA \wedge A) \\ &= g[F, \epsilon]. \end{aligned} \tag{1.100}$$

So F transforms covariantly under gauge transformations, i.e. depends only on ϵ and not $d\epsilon$. That should remind us of something, namely curvature.
In general relativity, under a co-ordinate transformation

$$g_{ab} \to g_{a'b'} = A_{a'}{}^a A_{b'}{}^b g_{ab}, \quad \Gamma^a{}_{bc} \to \Gamma^{a'}{}_{b'c'} = A^{a'}{}_a A_{b'}{}^b A_{c'}{}^c \Gamma^a{}_{bc} + \ldots, \tag{1.101}$$

where the remaining terms contain derivatives of A. The Riemann tensor $R^a{}_{bcd}$ contains derivatives of Γ and squared Γ terms, so one would expect second derivatives of A or squared first derivatives to appear. But

$$R^{a'}{}_{b'c'd'} = A^{a'}{}_a A_{b'}{}^b A_{c'}{}^c A_{d'}{}^d R^a{}_{bcd} \tag{1.102}$$

with no such $\partial A, \partial\partial A$ terms. F in Yang-Mills theory has the same property, that is not a coincidence. In general relativity,

$$[\nabla_a \nabla_b - \nabla_b \nabla_a] V_c = R_{abc}{}^d V_d. \tag{1.103}$$

The curvature is the commutator of two covariant derivatives. The same is true in Yang-Mills theory (see later).
We first go back to Maxwell's equations; the other half of these equations is (in vacuum)

$$d * F = 0. \tag{1.104}$$

The obvious generalisation of this is the Yang-Mills equation

$$D_A(*F) = 0, \tag{1.105}$$

that is in components,

$$\nabla_a F^{ab\alpha} + \frac{1}{6} g c_{\beta\gamma}{}^{\alpha} A_c^{\beta} F_{de}^{\gamma} \varepsilon^{bcde} = 0. \tag{1.106}$$

Lect. 5

Let us now calculate $D_A D_A X$, where X is a p-form in the adjoint representation of G. Then $Y = D_A X$ is a $(p+1)$-form, so we have

$$D_A Y = dY + gA \wedge Y + g(-)^p Y \wedge A, \quad Y = D_A X = dX + gA \wedge X + g(-)^{p+1} X \wedge A. \tag{1.107}$$

Then

$$\begin{aligned}
D_A D_A X &= d(dX + gA \wedge X + g(-)^{p+1} X \wedge A) + gA \wedge (dX + gA \wedge X + g(-)^{p+1} X \wedge A) \\
&\quad + g(-)^p (dX + gA \wedge X + g(-)^{p+1} X \wedge A) \wedge A \\
&= gdA \wedge X - gA \wedge dX + g(-)^{p+1} dX \wedge A - gX \wedge dA + gA \wedge dX + g^2 A \wedge A \wedge X \\
&\quad + g^2 (-)^{p+1} A \wedge X \wedge A + g(-)^p dX \wedge A + g^2 (-)^p A \wedge X \wedge A - g^2 X \wedge A \wedge A \\
&= g(dA + gA \wedge A) \wedge X - gX \wedge (dA + gA \wedge A) \\
&= g[F, X]
\end{aligned} \tag{1.108}$$

since F is a two-form.
That is exactly what you would expect from a curvature. F is often called the curvature form in mathematics (or Yang-Mills field strength in physics). So F must be the curvature of something, so you should think of A as being a connection one-form (in the mathematics world).

2 Connections and General Relativity

2.1 Vielbein Formalism

You should wonder whether the same ideas work in general relativity. In general relativity, everything involves just the metric tensor g_{ab}. All of the geometry of spacetime will be encoded into a line element

$$ds^2 = g_{ab} dx^a dx^b. \tag{2.1}$$

We try to extend this idea: g_{ab} is a d-dimensional metric with t timelike directions. That means in practice that you can always construct normal co-ordinates such that

$$\eta \sim \left((-)^t, (-)^{d-t} \right). \tag{2.2}$$

One can only do this at a point. But as η describes the tangent space of the manifold, we can rewrite the metric as

$$g_{ab} = e_a{}^\mu e_b{}^\nu \eta_{\mu\nu}, \quad \eta_{\mu\nu} = \mathrm{diag}\left((-)^t, (-)^{d-t} \right). \tag{2.3}$$

The objects $e_a{}^\mu$ are called **vierbein** or **vielbein fields** in general relativity, or **frame fields** in the mathematics world. It is not entirely obvious that you can always do this construction. At each point, g is a symmetric matrix, so can diagonalise it:

$$g = O^T D O, \quad D = \sum_i \lambda_i (f_i \otimes f_i), \; f_i = (0, \ldots, \underbrace{1}_{i\text{th place}}, 0, \ldots, 0). \tag{2.4}$$

There will be d non-zero eigenvalues λ_i, of which t will be negative and $d-t$ will be positive. Then by rescaling the eigenvectors, it should be clear that one can get g to the above form.
But while g_{ab} has $\frac{1}{2}d(d+1)$ components, $e_a{}^\mu$ has d^2 components, so many more. But Lorentz transformations

$$V^\mu \to V^\nu = \Lambda^\nu{}_\mu V^\mu \tag{2.5}$$

preserve the Lorentz metric:

$$\Lambda^T \eta \Lambda = \eta, \quad \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}. \tag{2.6}$$

In the general case, $\Lambda \in SO(d-t,t)$, and it is often useful to restrict attention to the component connected to the identity. One would not call this a Lorentz transformation, but a **generalised rotation**.
Under a (local) transformation of the frame fields,

$$e_a{}^\mu \to \tilde{e}_a{}^\mu = \Lambda^\mu{}_\nu(x) e_a{}^\nu, \tag{2.7}$$

the metric is left invariant:

$$g_{ab} \to \tilde{g}_{ab} = \tilde{e}_a{}^\mu \tilde{e}_b{}^\nu \eta_{\mu\nu} = \Lambda^\mu{}_\rho(x) e_a{}^\rho \Lambda^\nu{}_\sigma(x) e_b{}^\sigma \eta_{\mu\nu} = e_a{}^\rho e_b{}^\sigma \eta_{\rho\sigma} = g_{ab}. \tag{2.8}$$

You have found a new local invariance. We have enlarged the symmetry of general relativity (or ...) to be

a) general co-ordinate transformations,

b) local generalised rotations.

One needs the frame fields to describe fermions in general relativity. Greek indices $\mu, \nu, \ldots$ are **tangent space** indices (Lorentz indices), Latin indices $a, b, c, \ldots$ are **spacetime** indices.
g_{ab}, g^{ab} can raise and lower spacetime indices; $\eta_{\mu\nu}, \eta^{\mu\nu}$ can raise and lower tangent space indices:

$$e_{a\mu} = \eta_{\mu\nu} e_a{}^\nu, \quad e_a{}^\mu = \eta^{\mu\nu} e_{a\nu}, \quad \text{etc.} \tag{2.9}$$

We can now write

$$g_{ab} = e_a{}^\mu e_b{}^\nu = e_a{}^\mu e_{b\mu}. \tag{2.10}$$

The objects $e_a{}^\mu$ can also be used to convert spacetime vectors (tensors) into tangent space vectors (tensors):

$$V^a \to V^\mu = e_a{}^\mu V^a, \quad V^a = e^a{}_\mu V^\mu, \tag{2.11}$$

and indeed

$$e^a{}_\mu V^\mu = e^a{}_\mu e_b{}^\mu V^b = \delta^a{}_b V^b = V^a. \tag{2.12}$$

This works similarly for general tensors of type (r, s).
The next thing is some idea of a derivative: A covariant derivative is

$$V^a \to \nabla_b V^a, \tag{2.13}$$

such that under a co-ordinate transformation, if $V^{a'} = A^{a'}{}_a V^a$,

$$\nabla_b V^a \to \nabla_{b'} V^{a'} = A_{b'}{}^b A^{a'}{}_a \nabla_b V^a \tag{2.14}$$

with no derivatives of A, which are cancelled by the usual Christoffel symbols. What is the covariant derivative of $e_a{}^\mu$? It should be a $(0, 2)$ spacetime tensor, and a Lorentz vector. Under a Lorentz transformation $e \to e\Lambda$, one will normally get $\partial e \sim (\partial e)\Lambda + e\partial\Lambda$, so we need to add an extra term:

$$\nabla_b e_a{}^\mu = \partial_b e_a{}^\mu - \Gamma_b{}^c{}_a e_c{}^\mu + \omega_b{}^\mu{}_\sigma e_a{}^\sigma, \tag{2.15}$$

where $\omega_b{}^\mu{}_\sigma$ is the **spin connection**. The spin connection is needed to absorb the terms involving $\partial\Lambda$ if one performs a Lorentz transformation.
In Riemannian geometry and general relativity, one is accustomed to making a certain choice of connection, such that

$$\nabla_a g_{bc} = 0. \tag{2.16}$$

One wants to make an analogous choice for frame fields, which is consistent with it. The simplest way to arrange this is to make

$$\nabla_b e_a{}^\mu = 0, \quad \nabla_a \eta_{\mu\nu} = 0. \tag{2.17}$$

We can turn

$$\partial_b e_a{}^\mu - \Gamma_b{}^c{}_a e_c{}^\mu + \omega_b{}^\mu{}_\sigma e_a{}^\sigma = 0 \tag{2.18}$$

into an expression for the spin connection by multiplying by $e^a{}_\lambda$:

$$\omega_b{}^\mu{}_\lambda = \omega_b{}^\mu{}_\sigma e_a{}^\sigma e^a{}_\lambda = -e^a{}_\lambda \partial_b e_a{}^\mu + \Gamma_b{}^c{}_a e_c{}^\mu e^a{}_\lambda. \tag{2.19}$$

We can regard this as a definition of the spin connection (almost). This definition of the spin connection contains more information than Γ, so ω and Γ are <u>not</u> equivalent. Remember that a **metric connection** consists of two pieces:

$$\Gamma_b{}^c{}_a = \Gamma_{(b}{}^c{}_{a)} + \Gamma_{[b}{}^c{}_{a]}, \tag{2.20}$$

where the symmetric part is given by the Christoffel symbols and the antisymmetric part defines the **torsion**:

$$T_b{}^c{}_a = 2\Gamma_{[b}{}^c{}_{a]}. \tag{2.21}$$

We have not yet looked at $\nabla_a \eta_{\mu\nu} = 0$. Lect. 6

You can obtain an equation analogous to the one above by writing out $\nabla_b {e^a}_\nu = 0$:

$$\partial_b {e^a}_\nu + {\Gamma_b}^a{}_c {e^c}_\nu + \omega_{b\nu\sigma} e^{a\sigma} = 0. \tag{2.22}$$

Then multiply this by ${e_a}^\lambda$ to get

$$\omega_{b\nu\lambda} = -{e_a}^\lambda \partial_b {e^a}_\nu - {e_a}^\lambda {\Gamma_b}^a{}_c {e^c}_\nu. \tag{2.23}$$

This definition is equivalent to the one above.

You see that calculations like these are rather messy. Cartan called this a "debauch of indices". The point of using forms is to get rid of the indices. We still need to look at

$$\nabla_a \eta_{\mu\nu} = 0. \tag{2.24}$$

This gives

$$0 \stackrel{!}{=} \nabla_a \eta_{\mu\nu} = \partial_a \eta_{\mu\nu} + {\omega_{a\mu}}^\sigma \eta_{\sigma\nu} + {\omega_{a\nu}}^\sigma \eta_{\mu\sigma} = \omega_{a\mu\nu} + \omega_{a\nu\mu}. \tag{2.25}$$

Hence a spin connection that is metric is antisymmetric on its Lorentz indices.

So this how a spin connection is defined, but you really do not want to do it this way in practice.

Let us start again, remembering that a conncetion can have torsion as well as curvature. We demanded that

$$0 = \nabla_a g_{bc} = \partial_a g_{bc} - {\Gamma_a}^d{}_b g_{dc} - {\Gamma_a}^d{}_c g_{bd}. \tag{2.26}$$

For a symmetric connection, you can solve this in terms of Γ and discover that a symmetric metric connection is unique. No such luck for us!

Let us try to repeat the usual calculation with nonvanishing torsion. One starts with

$$0 = \nabla_a g_{bc} + \nabla_b g_{ca} - \nabla_c g_{ab}, \tag{2.27}$$

which gives

$$\begin{aligned}
\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab} &= {\Gamma_a}^d{}_b g_{dc} + {\Gamma_a}^d{}_c g_{bd} + {\Gamma_b}^d{}_c g_{da} + {\Gamma_b}^d{}_a g_{cd} - {\Gamma_c}^d{}_a g_{db} - {\Gamma_c}^d{}_b g_{ad} \\
&= 2\Gamma_{(a}{}^d{}_{b)} g_{dc} + 2\Gamma_{[a}{}^d{}_{c]} g_{bd} + 2\Gamma_{[b}{}^d{}_{c]} g_{da} \\
&= 2\Gamma_{(a}{}^d{}_{b)} g_{dc} + T_{[a}{}^d{}_{c]} g_{bd} + T_{[b}{}^d{}_{c]} g_{da}.
\end{aligned} \tag{2.28}$$

If the torsion vanishes, you can get what you are used to.

Let us recall a formula for the **curvature**:

$$R^a{}_{bcd} = \partial_c {\Gamma_d}^a{}_b - \partial_d {\Gamma_c}^a{}_b + {\Gamma_c}^a{}_e {\Gamma_d}^e{}_b - {\Gamma_d}^a{}_e {\Gamma_c}^e{}_b \tag{2.29}$$

or equivalently, by commuting covariant derivatives

$$\left(\nabla_c \nabla_d - \nabla_d \nabla_c\right) Z^a = R^a{}_{bcd} Z^b - {T_c}^e{}_d \nabla_e Z^a. \tag{2.30}$$

We have another connection which will have a curvature too:

$$\begin{aligned}
(\nabla_c \nabla_d - \nabla_d \nabla_c) V^\mu &= \nabla_c (\partial_d V^\mu + {\omega_d}^\mu{}_\sigma V^\sigma) - \nabla_d (\partial_c V^\mu + {\omega_c}^\mu{}_\sigma V^\sigma) \\
&= \partial_c \partial_d V^\mu + (\partial_c {\omega_d}^\mu{}_\sigma) V^\sigma + {\omega_d}^\mu{}_\sigma \partial_c V^\sigma - {\Gamma_c}^e{}_d (\partial_e V^\mu + {\omega_e}^\mu{}_\sigma V^\sigma) \\
&\quad + {\omega_c}^\mu{}_\lambda (\partial_d V^\lambda + {\omega_d}^\lambda{}_\sigma V^\sigma) - (c \leftrightarrow d),
\end{aligned} \tag{2.31}$$

we would like this to be a spin curvature and a torsion term. We look at the terms involving V and no derivatives of V, and identify

$$R_{cd}{}^{\mu}{}_{\sigma} = \partial_c \omega_d{}^{\mu}{}_{\sigma} - \partial_d \omega_c{}^{\mu}{}_{\sigma} + \omega_c{}^{\mu}{}_{\lambda} \omega_d{}^{\lambda}{}_{\sigma} - \omega_d{}^{\mu}{}_{\lambda} \omega_c{}^{\lambda}{}_{\sigma} \tag{2.32}$$

as the curvature of the spin connection. The remaining terms are

$$(-\Gamma_c{}^e{}_d + \Gamma_d{}^e{}_c)(\partial_e V^{\mu} + \omega_e{}^{\mu}{}_{\sigma} V^{\sigma}) = -T_c{}^e{}_d \nabla_e V^{\mu}. \tag{2.33}$$

We obtain the same form as before. Manipulations on Lorentz indices are analogous to manipulations on spacetime indices. Another fact which is almost miraculous is

$$R_{ab}{}^{\mu}{}_{\nu} = e_c{}^{\mu} e^d{}_{\nu} R_{ab}{}^c{}_d, \tag{2.34}$$

where the term on the left-hand side is the curvature from the spin connection, and the Riemann tensor on the right-hand side is the curvature from the Γ connection. This is true including torsion, but not for a non-metric connection.
This is not entirely obvious. If you like, you can prove it explicitly using a metric connection.

2.2 Form Notation

Everybody who has ever calculated $R_{ab}{}^c{}_d$ explicitly knows that it is a nightmare. All these expressions look much easier when written in terms of forms.
We start with a basis of ("pseudo-orthonormal") one-forms

$$E^{\mu} = e_a{}^{\mu} dx^a. \tag{2.35}$$

This is enough to specify the metric by

$$\eta_{\mu\nu} E^{\mu} \otimes E^{\nu} = \eta_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu} dx^a \otimes dx^b. \tag{2.36}$$

Since $\{E^{\mu}\}$ form a basis, one can use them in any practical calculation. There is, in addition, a connection one-form built from the spin connection

$$\omega_{\mu\nu} = -\omega_{\nu\mu} = \omega_{a\mu\nu} dx^a. \tag{2.37}$$

We can define a torsion two-form

$$T^{\lambda} = e_a{}^{\lambda} \frac{1}{2} T_b{}^a{}_c dx^b \wedge dx^c. \tag{2.38}$$

Lastly, there is a curvature two-form

$$\mathcal{R}^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu cd} dx^c \wedge dx^d. \tag{2.39}$$

These forms contain all the information you could possibly want. Now we will translate everything into this language. No sane person, after they have seen this, will do calculations any other way.

We discover that

$$\begin{aligned}
dE^\mu + \omega^\mu{}_\nu \wedge E^\nu &= d(e_a{}^\mu dx^a) + \omega_a{}^\mu{}_\nu dx^a \wedge e_b{}^\nu dx^b \\
&= (\partial_b e_a{}^\mu) dx^b \wedge dx^a + \omega_a{}^\mu{}_\nu e_b{}^\nu dx^a \wedge dx^b \\
&= \frac{1}{2}(\partial_a e_b{}^\mu - \partial_b e_a{}^\mu + \omega_a{}^\mu{}_\nu e_b{}^\nu - \omega_b{}^\mu{}_\nu e_a{}^\nu) dx^a \wedge dx^b \\
&= \frac{1}{2}(\Gamma_a{}^c{}_b e_c{}^\mu - \omega_a{}^\mu{}_\sigma e_b{}^\sigma - \Gamma_b{}^c{}_a e_c{}^\mu + \omega_b{}^\mu{}_\sigma e_a{}^\sigma + \omega_a{}^\mu{}_\nu e_b{}^\nu - \omega_b{}^\mu{}_\nu e_a{}^\nu) dx^a \wedge dx^b \\
&= \frac{1}{2} T_a{}^c{}_b e_c{}^\mu dx^a \wedge dx^b = T^\mu, \qquad (2.40)
\end{aligned}$$

where we have used

$$\partial_b e_{a\nu} = \Gamma_b{}^c{}_a e_{c\nu} - \omega_{b\nu\sigma} e_a{}^\sigma. \qquad (2.41)$$

This is **Cartan's first equation of structure**:

$$dE^\mu + \omega^\mu{}_\nu \wedge E^\nu = T^\mu. \qquad (2.42)$$

What about the curvature? You can substitute in the components to see that **Cartan's second equation of structure** holds:

$$d\omega^{\lambda\sigma} + \omega^\lambda{}_\nu \wedge \omega^{\nu\sigma} = \mathcal{R}^{\lambda\sigma}. \qquad (2.43)$$

Lect. 7

The second equation of structure is very similar to Yang-Mills theory, where $F = dA + A \wedge A$, except that

$$\begin{cases} F, A & \text{take values in the adjoint representation of a gauge group,} \\ R, \omega & \text{take values in the Lorentz group (or whatever stands in for it).} \end{cases} \qquad (2.44)$$

However, in Yang-Mills theory there is no analogue of torsion T or vielbeins E. This leads to problems if you try to interpret general relativity as a Yang-Mills theory for the Lorentz group. You could write something like

$$\mathcal{R} = D_\omega \omega, \qquad (2.45)$$

but we will not use this notation.
Let us look at **Bianchi identites**. The first identity is obtained by taking d of Cartan's first equation of structure:

$$\begin{aligned}
dT^\mu &= d(dE^\mu + \omega^\mu{}_\nu \wedge E^\nu) \\
&= d\omega^\mu{}_\nu \wedge E^\nu - \omega^\mu{}_\nu \wedge dE^\nu \\
&= (\mathcal{R}^\mu{}_\nu - \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu) \wedge E^\nu - \omega^\mu{}_\nu \wedge (T^\nu - \omega^\nu{}_\sigma \wedge E^\sigma) \\
&= \mathcal{R}^\mu{}_\nu \wedge E^\nu - \omega^\mu{}_\nu \wedge T^\nu. \qquad (2.46)
\end{aligned}$$

For vanishing torsion, as in general relativity, one has

$$\mathcal{R}^\lambda{}_\mu \wedge E^\mu = 0, \qquad (2.47)$$

which corresponds to the usual $R^a{}_{[bcd]} = 0$.
There is a second Bianchi identity: Take d of the defintion of curvature.

$$\begin{aligned}
\mathcal{R}^\lambda{}_\mu &= d(d\omega^\lambda{}_\mu + \omega^\lambda{}_\rho \wedge \omega^\rho{}_\mu) \\
&= d\omega^\lambda{}_\rho \wedge \omega^\rho{}_\mu - \omega^\lambda{}_\rho \wedge d\omega^\rho{}_\mu \\
&= (\mathcal{R}^\lambda{}_\rho - \omega^\lambda{}_\nu \wedge \omega^\nu{}_\rho) \wedge \omega^\rho{}_\mu - \omega^\lambda{}_\rho \wedge (\mathcal{R}^\rho{}_\mu - \omega^\rho{}_\nu \wedge \omega^\nu{}_\mu) \\
&= \mathcal{R}^\lambda{}_\rho \wedge \omega^\rho{}_\mu - \omega^\lambda{}_\rho \wedge \mathcal{R}^\rho{}_\mu. \qquad (2.48)
\end{aligned}$$

We could write this as

$$d\mathcal{R} = [\mathcal{R}, \omega], \qquad (2.49)$$

which is again very similar to Yang-Mills theory. In components, this is $\nabla_{[a}\mathcal{R}^\lambda{}_{|\mu|bc]} = 0$.

2.3 Explicit Example

If one wants to evaluate the curvature and use it for something, then using this formalism is relatively easy. In general relativity, the torsion vanishes, $T^\mu = 0$ (typically). Then you can use

$$0 = dE^\mu + \omega^\mu{}_\nu \wedge E^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \qquad (2.50)$$

to find a metric connection $\omega^\mu{}_\nu$ for a given (pseudo-)orthonormal basis of one-forms E^μ. You can expand the two-form dE^λ as

$$dE^\lambda = \frac{1}{2}c^\lambda{}_{\mu\rho}E^\mu \wedge E^\rho = -\omega^\lambda{}_\mu \wedge E^\mu \qquad (2.51)$$

and invert this relation to get

$$\omega_{\mu\nu} = \frac{1}{2}(-c_{\lambda\mu\nu} - c_{\mu\lambda\nu} + c_{\nu\lambda\mu})E^\lambda. \qquad (2.52)$$

This defines the connection one-form. Then you can use the second equation of structure to find the curvature.
Most of the time, you can actually find ω by inspection, without using this formula.
Example: Spherically symmetric static spacetimes with line element

$$ds^2 = -V^2(r)dt^2 + W(r)^2dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 = \eta_{\mu\nu}E^\mu \otimes E^\nu \qquad (2.53)$$

which defines an orthonormal basis of one-forms:

$$E^0 = V(r)dt, \quad E^1 = W(r)dr, \quad E^2 = r\,d\theta, \quad E^3 = r\,\sin\theta\,d\phi. \qquad (2.54)$$

You can think of the coefficients in these expressions as $e_a{}^\mu$. This relates the basis $\{E^\mu\}$ to a co-ordinate basis; it is useful to invert this:

$$dt = \frac{E^0}{V(r)}, \quad dr = \frac{E^1}{W(r)}, \quad d\theta = \frac{E^2}{r}, \quad d\phi = \frac{E^3}{r\,\sin\theta}. \qquad (2.55)$$

The coefficients appearing here form the matrix $e^a{}_\mu$. The first equation of structure gives

$$\begin{aligned} dE^0 &= d(V(r)dt) \\ &= -V'(r)dt \wedge dr \\ &= -\frac{V'(r)}{V(r)W(r)} E^0 \wedge E^1 \\ &= -\omega^0{}_1 \wedge E^1 - \omega^0{}_2 \wedge E^2 - \omega^0{}_3 \wedge E^3, \end{aligned} \tag{2.56}$$

since $\omega^0{}_0 = -\omega_{00} = 0$. This means that $\omega^0{}_2$ is proportional to E^2, $\omega^0{}_3$ is proportional to E^3 and

$$\omega^0{}_1 = \frac{V'(r)}{V(r)W(r)} E^0 + \alpha E^1. \tag{2.57}$$

Similarly,

$$\begin{aligned} dE^1 &= d(W(r)dr) \\ &= 0 \\ &= \omega^1{}_0 \wedge E^0 - \omega^1{}_2 \wedge E^2 - \omega^1{}_3 \wedge E^3. \end{aligned} \tag{2.58}$$

We use $\omega^1{}_0 = \omega_{10} = -\omega_{01} = \omega^0{}_1$ to see there is no E^1 term and hence

$$\omega^0{}_1 = \frac{V'(r)}{V(r)W(r)} E^0. \tag{2.59}$$

$\omega^1{}_2$ is proportional to E^2 and $\omega^1{}_3$ is proportional to E^3.

$$\begin{aligned} dE^2 &= d(rd\theta) \\ &= dr \wedge d\theta \\ &= \frac{1}{rW(r)} E^1 \wedge E^2 \\ &= \omega^2{}_0 \wedge E^0 - \omega^2{}_1 \wedge E^1 - \omega^2{}_3 \wedge E^3. \end{aligned} \tag{2.60}$$

Using that $\omega^2{}_0 = \omega_{20} = -\omega_{02} = \omega^0{}_2$ and $\omega^2{}_1 = \omega_{21} = -\omega_{12} = -\omega^1{}_2$, we now have

$$\omega^0{}_2 = 0, \quad \omega^1{}_2 = -\frac{1}{rW(r)} E^2. \tag{2.61}$$

Finally,

$$\begin{aligned} dE^3 &= d(r \sin\theta d\phi) \\ &= \sin\theta dr \wedge d\phi + r\cos\theta d\theta \wedge d\phi \\ &= \frac{1}{rW(r)} E^1 \wedge E^3 + \frac{1}{r\tan\theta} E^2 \wedge E^3 \\ &= \omega^3{}_0 \wedge E^0 - \omega^3{}_1 \wedge E^1 - \omega^3{}_2 \wedge E^2. \end{aligned} \tag{2.62}$$

Using $\omega^3{}_0 = \omega^0{}_3$, $\omega^3{}_1 = -\omega^1{}_3$ and $\omega^3{}_2 = -\omega^2{}_3$ we obtain

$$\omega^0{}_3 = 0, \quad \omega^1{}_3 = -\frac{1}{rW(r)} E^3, \quad \omega^2{}_3 = -\frac{1}{r\tan\theta} E^3. \tag{2.63}$$

We can summarise this in a table:

$\omega^a{}_b$	$b=0$	$b=1$	$b=2$	$b=3$
$a=0$	0	$\frac{V'(r)}{V(r)W(r)}E^0$	0	0
$a=1$	$\frac{V'(r)}{V(r)W(r)}E^0$	0	$-\frac{1}{rW(r)}E^2$	$-\frac{1}{rW(r)}E^3$
$a=2$	0	$\frac{1}{rW(r)}E^2$	0	$-\frac{1}{r\tan\theta}E^3$
$a=3$	0	$\frac{1}{rW(r)}E^3$	$\frac{1}{r\tan\theta}E^3$	0

There are always $\frac{1}{2}d(d-1)$ nontrivial components of ω. Compare this with $\Gamma_b{}^a{}_c$, which has $\frac{1}{2}d^2(d+1)$ components for a symmetric connection. Lect. 8

Now we will calculate the curvature two-form using

$$\mathcal{R}^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu, \tag{2.64}$$

in the basis of two-forms given by $E^\rho \wedge E^\sigma$, where

$$\mathcal{R}^\mu{}_\nu = \frac{1}{2}\mathcal{R}^\mu{}_{\nu\rho\sigma}E^\rho \wedge E^\sigma. \tag{2.65}$$

Since

$$\mathcal{R}_{\mu\nu} = -\mathcal{R}_{\nu\mu}, \tag{2.66}$$

we again only need to calculate six components:

$$\begin{aligned}
\mathcal{R}^0{}_1 &= d\omega^0{}_1 + \omega^0{}_\mu \wedge \omega^\mu{}_1 \\
&= d\omega^0{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 \\
&= \left(\frac{V''(r)}{V(r)W(r)} - \frac{V'^2(r)}{V^2(r)W(r)} - \frac{V'(r)W'(r)}{V(r)W^2(r)}\right) dr \wedge E^0 + \frac{V'(r)}{V(r)W(r)} dE^0 \\
&= \left(-\frac{V''(r)}{V(r)W^2(r)} + \frac{V'^2(r)}{V^2(r)W^2(r)} + \frac{V'(r)W'(r)}{V(r)W^3(r)}\right) E^0 \wedge E^1 - \frac{V'^2(r)}{V^2(r)W^2(r)} E^0 \wedge E^1 \\
&= \frac{1}{W^2(r)}\left(-\frac{V''(r)}{V(r)} + \frac{V'(r)W'(r)}{V(r)W(r)}\right) E^0 \wedge E^1.
\end{aligned} \tag{2.67}$$

$$\begin{aligned}
\mathcal{R}^0{}_2 &= d\omega^0{}_2 + \omega^0{}_1 \wedge \omega^1{}_2 + \omega^0{}_3 \wedge \omega^3{}_2 \\
&= \frac{V'(r)}{V(r)W(r)} E^0 \wedge \left(-\frac{1}{rW(r)}E^2\right) \\
&= -\frac{V'(r)}{rV(r)W^2(r)} E^0 \wedge E^2.
\end{aligned} \tag{2.68}$$

$$\begin{aligned}
\mathcal{R}^0{}_3 &= d\omega^0{}_3 + \omega^0{}_1 \wedge \omega^1{}_3 + \omega^0{}_2 \wedge \omega^2{}_3 \\
&= \frac{V'(r)}{V(r)W(r)} E^0 \wedge \left(-\frac{1}{rW(r)}E^3\right) \\
&= -\frac{V'(r)}{rV(r)W^2(r)} E^0 \wedge E^3.
\end{aligned} \tag{2.69}$$

$$\begin{aligned}
\mathcal{R}^1{}_2 &= d\omega^1{}_2 + \omega^1{}_0 \wedge \omega^0{}_2 + \omega^1{}_3 \wedge \omega^3{}_2 \\
&= d\left(-\frac{1}{rW(r)}E^2\right) + \left(-\frac{1}{rW(r)}E^3\right) \wedge \left(\frac{1}{r\cot\theta}E^3\right) \\
&= \left(\frac{1}{r^2W(r)} + \frac{W'(r)}{rW^2(r)}\right) dr \wedge E^2 - \frac{1}{rW(r)}dE^2 \\
&= \frac{1}{W^2(r)}\left(\frac{1}{r^2} + \frac{W'(r)}{rW(r)}\right) E^1 \wedge E^2 - \frac{1}{r^2W^2(r)}E^1 \wedge E^2 \\
&= \frac{W'(r)}{rW^3(r)}E^1 \wedge E^2. \qquad (2.70)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^1{}_3 &= d\omega^1{}_3 + \omega^1{}_0 \wedge \omega^0{}_3 + \omega^1{}_2 \wedge \omega^2{}_3 \\
&= d\left(-\frac{1}{rW(r)}E^3\right) + \frac{1}{rW(r)} \cdot \frac{1}{r\tan\theta}E^2 \wedge E^3 \\
&= \left(\frac{1}{r^2W(r)} + \frac{W'(r)}{rW^2(r)}\right) dr \wedge E^3 - \frac{1}{rW(r)}\left(\frac{1}{rW(r)}E^1 \wedge E^3\right. \\
&\quad \left. + \frac{1}{r\tan\theta}E^2 \wedge E^3\right) + \frac{1}{r^2W(r)\tan\theta}E^2 \wedge E^3 \\
&= \frac{1}{W^2(r)}\left(\frac{1}{r^2} + \frac{W'(r)}{rW(r)}\right) E^1 \wedge E^3 - \frac{1}{r^2W^2(r)}E^1 \wedge E^3 \\
&= \frac{W'(r)}{rW^3(r)}E^1 \wedge E^3. \qquad (2.71)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^2{}_3 &= d\omega^2{}_3 + \omega^2{}_0 \wedge \omega^0{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 \\
&= d\left(-\frac{1}{r\tan\theta}E^3\right) - \frac{1}{rW(r)} \cdot \frac{1}{rW(r)}E^2 \wedge E^3 \\
&= \frac{1}{r^2\tan\theta}dr \wedge E^3 + \frac{1}{r\sin^2\theta}d\theta \wedge E^3 - \frac{1}{r\tan\theta}\left(\frac{1}{rW(r)}E^1 \wedge E^3 + \frac{1}{r\tan\theta}E^2 \wedge E^3\right) \\
&\quad -\frac{1}{r^2W^2(r)}E^2 \wedge E^3 \\
&= \left(\frac{1}{r^2W(r)\tan\theta} - \frac{1}{r^2W(r)\tan\theta}\right) E^1 \wedge E^2 + \left(\frac{1}{r^2\sin^2\theta} - \frac{1}{r^2\tan^2\theta} - \frac{1}{r^2W^2(r)}\right) E^2 \wedge E^3 \\
&= \frac{1}{r^2}\left(1 - \frac{1}{W^2(r)}\right) E^2 \wedge E^3. \qquad (2.72)
\end{aligned}$$

Now in practice you want to calculate solutions of Einstein's equations. For this you need to calculate the Ricci tensor, defined by

$$\mathcal{R}^\mu{}_{\nu\mu\sigma} = R_{\nu\sigma}. \qquad (2.73)$$

We can read off the components $\mathcal{R}^\mu{}_{\nu\mu\sigma}$ from the expressions above, e.g.

$$R^0{}_{101} = \frac{1}{W^2(r)}\left(-\frac{V''(r)}{V(r)} + \frac{V'(r)W'(r)}{V(r)W(r)}\right), \qquad (2.74)$$

noticing that they are only non-vanishing if $\nu = \sigma$, which is a consequence of the symmetry of the problem. It follows that we must have

$$R_{\nu\sigma} = 0, \quad \nu \neq \sigma. \qquad (2.75)$$

The diagonal components are

$$
\begin{aligned}
R_{00} &= R^1{}_{010} + R^2{}_{020} + R^3{}_{030} \\
&= \frac{1}{W^2(r)}\left(\frac{V''(r)}{V(r)} - \frac{V'(r)W'(r)}{V(r)W(r)}\right) + \frac{1}{W^2(r)}\frac{V'(r)}{rV(r)} + \frac{1}{W^2(r)}\frac{V'(r)}{rV(r)} \\
&= \frac{1}{W^2(r)}\left(\frac{V''(r)}{V(r)} - \frac{V'(r)W'(r)}{V(r)W(r)} + 2\frac{V'(r)}{rV(r)}\right). \qquad (2.76)
\end{aligned}
$$

$$
\begin{aligned}
R_{11} &= R^0{}_{101} + R^2{}_{121} + R^3{}_{131} \\
&= \frac{1}{W^2(r)}\left(-\frac{V''(r)}{V(r)} + \frac{V'(r)W'(r)}{V(r)W(r)} + 2\frac{W'(r)}{rW(r)}\right). \qquad (2.77)
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= R^0{}_{202} + R^1{}_{212} + R^3{}_{232} \\
&= \frac{1}{W^2(r)}\left(-\frac{V'(r)}{rV(r)} + \frac{W'(r)}{rW(r)} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \qquad (2.78)
\end{aligned}
$$

and $R_{33} = R_{22}$ (exercise). For vacuum solutions, we need $R_{\mu\nu} = 0$, so

$$0 = R_{00} + R_{11} = \frac{2}{rW^2(r)}\left(\frac{V'(r)}{V(r)} + \frac{W'(r)}{W(r)}\right), \qquad (2.79)$$

and hence

$$\log V(r)W(r) = \text{constant}, \quad V(r)W(r) = \text{constant}. \qquad (2.80)$$

If we demand that spacetime is flat as $r \to \infty$, it is natural to set

$$W(r) = \frac{1}{V(r)} \qquad (2.81)$$

Then

$$R_{22} = V^2(r)\left(-2\frac{V'(r)}{rV(r)} - \frac{1}{r^2}\right) + \frac{1}{r^2} \qquad (2.82)$$

and so we have to solve

$$2\frac{V'(r)}{V(r)} = -\frac{1}{r} + \frac{1}{rV^2(r)} = \frac{1 - V^2(r)}{rV^2(r)}. \qquad (2.83)$$

This is an ordinary differential equation that you can easily solve:

$$\int \frac{2V(r)\,dV}{1 - V^2(r)} = \int \frac{dr}{r}, \quad V^2(r) = 1 - \frac{\text{constant}}{r}. \qquad (2.84)$$

We have rediscovered the Schwarzschild solution. This is the easiest way to find solutions to Einstein's equations.

3 Integration

You have learned in general relativity that if we want to integrate a scalar ϕ over a d-dimensional domain D (with boundary ∂D), then

$$I = \int_D d^d x \sqrt{g}\, \phi(x) \tag{3.1}$$

is independent of the choice of co-ordinates, where

$$g = |\det g_{ab}| \tag{3.2}$$

and $\int d^d x$ is interpreted as a Riemann integral.
Now suppose that $\phi = \nabla_a V^a$, then (**Gauss' theorem**)

$$I = \int_D \nabla_a V^a = \int_{\partial D} d\Sigma_a\, V^a, \tag{3.3}$$

where $d\Sigma_a = n_a \cdot$(volume element of ∂D) for an outward unit normal n_a. The metric on ∂D is

$$h_{ab} = g_{ab} \pm n_a n_b, \tag{3.4}$$

where there is a plus if n is timelike and a minus if n is spacelike. Lect. 9
We will now replace the covariant volume element $d^d x \sqrt{g}$ in the above formulation by a **volume form**, the d-form

$$\epsilon = E^1 \wedge E^2 \wedge \ldots \wedge E^d, \tag{3.5}$$

where $\{E^\mu\}$ are a basis of orthonormal one-forms. Remember the alternating symbol (in the tangent space)

$$\eta^{\mu\nu\ldots\tau} = \begin{cases} +1 & (\mu\nu\ldots\tau) \text{ is an even permutation of } (1,\ldots,d) \\ -1 & (\mu\nu\ldots\tau) \text{ is an odd permutation of } (1,\ldots,d) \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

$\eta_{\mu\nu\ldots\tau}$ is found by lowering with the Lorentz metric (or more generally, the tangent space metric). Hence,

$$\eta_{\mu\nu\ldots\tau} = (-)^t \eta^{\mu\nu\ldots\tau}. \tag{3.7}$$

(Normally you should not write equations like this one.) In terms of components, we then have

$$\epsilon = \frac{1}{d!}(-)^t \eta_{\mu\nu\ldots\tau}\, E^\mu \wedge E^\nu \wedge \ldots \wedge E^\tau. \tag{3.8}$$

Expressing this in a co-ordinate basis, using $E^\mu = e_a{}^\mu dx^a$,

$$\begin{aligned} \epsilon &= \frac{1}{d!}(-)^t \eta_{\mu\nu\ldots\tau}\, e_a{}^\mu e_b{}^\nu \ldots e_f{}^\tau dx^a \wedge dx^b \wedge \ldots \wedge dx^f \\ &= \frac{1}{d!}(-)^t \eta_{\mu\nu\ldots\tau}\, e_a{}^\mu e_b{}^\nu \ldots e_f{}^\tau \eta^{abc\ldots f} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^d \\ &= (-)^t (\det e) d^d x. \end{aligned} \tag{3.9}$$

$\det e$ is almost the same as $\sqrt{g}$:

$$g_{ab} = e_a{}^\mu e_b{}^\nu \eta_{\mu\nu} \quad \Rightarrow \; \det g = \det(e^2 \eta) = \pm(\det e)^2. \tag{3.10}$$

Hence $|\det e| = \sqrt{|\det g|} = \sqrt{g}$. What you discover is that ϵ reproduces the previous expression for the volume element. So we define integration over a d-dimensional region to be

$$\int_D \phi \, d(Vol) \equiv \int_D \phi \, \epsilon. \tag{3.11}$$

One always integrates a d-form over a space of dimension d. Alternatively,

$$\int_D \phi \, d(Vol) \;\Rightarrow\; \int_D * \phi. \tag{3.12}$$

The first and most imporant result for integrals over forms is **Stokes' theorem**. We prove a pedestrian version, where the region D is bounded by two surfaces which can be taken as $\lambda = 0$ and $\lambda = 1$. We can then choose co-ordinates in D such that the metric is

$$ds^2 = d\lambda^2 + ds_\perp^2, \tag{3.13}$$

such that $g_{\lambda\lambda} = 1$ and $g_{\lambda i} = 0$ (these are Gaussian normal co-ordinates, see GR course). The volume form on D is $d\lambda \wedge d(Vol)_{d-1}$, where $d(Vol)_{d-1}$ is a volume form on surfaces $\lambda =$ constant. Now take a $(d-1)$-form which is proportional to $d(Vol)$, written as $f(\lambda, x^i) d(Vol)$, and integrate

$$\int_D \frac{\partial f(\lambda, x^i)}{\partial \lambda} d\lambda \wedge d(Vol) = \int_D df(\lambda, x^i) \wedge d(Vol) = \int_D d\left(f(\lambda, x^i) \wedge d(Vol)\right), \tag{3.14}$$

since $(\partial f / \partial x^i) dx^i \wedge d(Vol) = 0$ and $\partial d(Vol)/\partial \lambda = 0$. As a one-dimensional integral over λ this is

$$\int_D \frac{\partial f(\lambda, x^i)}{\partial \lambda} d\lambda \wedge d(Vol) = \int_{\partial D(\lambda=1)} f(1, x^i) d(Vol) - \int_{\partial D(\lambda=0)} f(0, x^i) d(Vol), \tag{3.15}$$

which is an integral over the boundary. So in this case we have

$$\int_D d\omega = \int_{\partial D} \omega. \tag{3.16}$$

This is by far the easiest version of Stokes' theorem. A corollary of this is

$$0 = \int_D ddg = \int_{\partial D} dg = \int_{\partial\partial D} g \tag{3.17}$$

for any $(d-2)$-form g. So the boundary of a boundary is empty.

3.1 Action for General Relativity

We want to give an action for general relativity in $d = 4$. You are probably used to the Einstein-Hilbert action

$$\int d^d x \, \sqrt{g} \, R; \tag{3.18}$$

from this you derive $R_{ab} = 0$ in vacuum, but under various assumptions on the connection. There is an alternative formulation which makes the requirements on the connection appear more natural:

$$I = \int_D (R^{\mu\nu}(\omega) \wedge E^\rho \wedge E^\sigma) \eta_{\mu\nu\rho\sigma} = \int_D \left(\left(d\omega^{\mu\lambda} + \omega^{\mu\nu} \wedge \omega_\nu{}^\lambda \right) \wedge E^\rho \wedge E^\sigma \right) \eta_{\mu\nu\rho\sigma}. \tag{3.19}$$

This action contains two types of fields: the vielbein fields and the connection ω. We require I to be stationary under arbitrary variation of both E and ω. Note that in this action, $\eta_{\mu\nu\rho\sigma}$ projects out the symmetric part of ω. We only need to consider the antisymmetric part of ω, so one automatically has a metric connection.
Vary E

$$\delta I = \int_D (R^{\mu\nu}(\omega) \wedge \delta E^\rho \wedge E^\sigma + R^{\mu\nu}(\omega) \wedge E^\rho \wedge \delta E^\sigma)\eta_{\mu\nu\rho\sigma} = \int_D 2(R^{\mu\nu}(\omega) \wedge \delta E^\rho \wedge E^\sigma)\eta_{\mu\nu\rho\sigma}; \tag{3.20}$$

if this is supposed to vanish for arbitrary δE^ρ, we must have

$$R^{\mu\nu} \wedge E^\rho \eta_{\mu\nu\rho\sigma} = 0. \tag{3.21}$$

In components, this is

$$\frac{1}{2} R^{\mu\nu}{}_{\lambda\tau} E^\lambda \wedge E^\tau \wedge E^\rho \eta_{\mu\nu\rho\sigma} = 0, \tag{3.22}$$

or

$$R^{\mu\nu}{}_{[\lambda\tau}\eta_{\rho]\mu\nu\sigma} = 0. \tag{3.23}$$

Contract this with $\eta^{\lambda\tau\rho\kappa}$, which does not annihilate any information in the equation:

$$\begin{aligned} 0 &= R^{\mu\nu}{}_{\lambda\tau}\eta_{\rho\mu\nu\sigma}\eta^{\lambda\tau\rho\kappa} \\ &= 2R^{\mu\nu}{}_{\lambda\tau}\left[\delta_\mu{}^\kappa \delta_\nu{}^\lambda \delta_\sigma{}^\tau + \delta_\mu{}^\lambda \delta_\nu{}^\tau \delta_\sigma{}^\kappa + \delta_\mu{}^\tau \delta_\nu{}^\kappa \delta_\sigma{}^\lambda\right] \\ &= 2\left(R^{\kappa\lambda}{}_{\lambda\tau}\delta_\sigma{}^\tau + R^{\lambda\tau}{}_{\lambda\tau}\delta_\sigma{}^\kappa + R^{\tau\kappa}{}_{\lambda\tau}\delta_\sigma{}^\lambda\right) \\ &= 2\left(-2R^\kappa{}_\sigma + R\delta_\sigma{}^\kappa\right), \end{aligned} \tag{3.24}$$

which you recognise as the vacuum Einstein equations. Contraction gives

$$R^\kappa{}_\sigma = 0, \tag{3.25}$$

so the Ricci tensor of the connection ω vanishes.
Now we try to vary ω:

$$\delta I = \int_D \left(\left(d\delta\omega^{\mu\lambda} + \delta\omega^{\mu\nu} \wedge \omega_\nu{}^\lambda + \omega^{\mu\nu} \wedge \delta\omega_\nu{}^\lambda\right) \wedge E^\rho \wedge E^\sigma\right)\eta_{\mu\nu\rho\sigma}. \tag{3.26}$$

This should vanish for arbitrary ω. Use the identity for a one-form X and a two-form Y

$$\int_{\partial D} X \wedge Y = \int_D d(X \wedge Y) = \int_D dX \wedge Y - X \wedge dY. \tag{3.27}$$

When varying something, you always have to worry about boundary conditions. Here we put $\delta\omega^{\mu\nu} = 0$ on the boundary. One could do this more generally, and consider boundary terms in the action as well (see Black Holes course).
Setting the boundary term to zero, we can turn $(d\delta\omega) \wedge E \wedge E$ into $\delta\omega \wedge d(E \wedge E)$:

$$\begin{aligned} \delta I &= \int_D \left\{\delta\omega^{\mu\lambda} \wedge (dE^\rho \wedge E^\sigma - E^\rho \wedge dE^\sigma) + 2\,(\delta\omega^{\mu\nu}) \wedge \omega_\nu{}^\lambda \wedge E^\rho \wedge E^\sigma\right\}\eta_{\mu\lambda\rho\sigma} \\ &= \int_D \delta\omega^{\mu\nu} \wedge \left((dE^\rho \wedge E^\sigma - E^\rho \wedge dE^\sigma)\,\eta_{\mu\nu\rho\sigma} + 2\omega_\nu{}^\lambda \wedge E^\rho \wedge E^\sigma \eta_{\mu\lambda\rho\sigma}\right). \end{aligned} \tag{3.28}$$

This should vanish for arbitrary $\omega^{\mu\nu}$, so we get

$$\left(dE^\rho \wedge E^\sigma - E^\rho \wedge dE^\sigma\right)\eta_{\mu\nu\rho\sigma} + 2\omega_{[\nu}{}^\lambda \wedge E^\rho \wedge E^\sigma \eta_{\mu]\lambda\rho\sigma} = 0. \tag{3.29}$$

Recall Cartan's first equation of structure and replace $dE^\rho = T^\rho - \omega^\rho{}_\tau \wedge E^\tau$:

$$\left((T^\rho - \omega^\rho{}_\tau \wedge E^\tau) \wedge E^\sigma - E^\rho \wedge (T^\sigma - \omega^\sigma{}_\tau \wedge E^\tau)\right)\eta_{\mu\nu\rho\sigma} + 2\omega_{[\nu}{}^\lambda \wedge E^\rho \wedge E^\sigma \eta_{\mu]\lambda\rho\sigma} = 0. \tag{3.30}$$

Now use

$$\begin{aligned}
& 2\omega_{[\nu}{}^\lambda \wedge E^\rho \wedge E^\sigma \eta_{\mu]\lambda\rho\sigma} + \left(-\omega^\rho{}_\tau \wedge E^\tau \wedge E^\sigma + E^\rho \wedge \omega^\sigma{}_\tau \wedge E^\tau\right)\eta_{\mu\nu\rho\sigma} \\
= \; & 2\omega_{[\nu}{}^\lambda \wedge E^\rho \wedge E^\sigma \eta_{\mu]\lambda\rho\sigma} - 2\omega^\rho{}_\lambda \wedge E^\lambda \wedge E^\sigma \eta_{\mu\nu\rho\sigma} \\
= \; & 2\left(\omega_{\tau[\nu}{}^\lambda \eta_{\mu]\lambda\rho\sigma} - \omega_\tau{}^\lambda{}_\rho \eta_{\mu\nu\lambda\sigma}\right) E^\tau \wedge E^\rho \wedge E^\sigma \\
= \; & 2\left(-\omega_\tau{}^\lambda{}_{[\nu} \eta_{\mu]\lambda\rho\sigma} - \omega_\tau{}^\lambda{}_\rho \eta_{\mu\nu\lambda\sigma}\right) E^\tau \wedge E^\rho \wedge E^\sigma,
\end{aligned} \tag{3.31}$$

where we expanded $\omega_\nu{}^\lambda = \omega_{\tau\nu}{}^\lambda E^\tau$ etc. Take the components of this three-form and multiply by $\eta^{\tau\rho\sigma\kappa}$, which is taking the Hodge dual:

$$\begin{aligned}
& 2\left(-\omega_\tau{}^\lambda{}_{[\nu} \eta_{\mu]\lambda\rho\sigma}\eta^{\tau\rho\sigma\kappa} - \omega_\tau{}^\lambda{}_\rho \eta_{\mu\nu\lambda\sigma}\eta^{\tau\rho\sigma\kappa}\right) \\
= \; & -\omega_\tau{}^\lambda{}_\nu \eta_{\rho\sigma\mu\lambda}\eta^{\rho\sigma\tau\kappa} + \omega_\tau{}^\lambda{}_\mu \eta_{\rho\sigma\nu\lambda}\eta^{\rho\sigma\tau\kappa} + 2\omega_\tau{}^\lambda{}_\rho \eta_{\sigma\mu\nu\lambda}\eta^{\sigma\tau\rho\kappa} \\
= \; & 4\omega_\tau{}^\lambda{}_\nu \delta_{[\mu}{}^\tau \delta_{\lambda]}{}^\kappa - 4\omega_\tau{}^\lambda{}_\mu \delta_{[\nu}{}^\tau \delta_{\lambda]}{}^\kappa - 12\omega_\tau{}^\lambda{}_\rho \delta_{[\mu}{}^\tau \delta_\nu{}^\rho \delta_{\lambda]}{}^\kappa \\
= \; & 4\omega_{[\mu}{}^\lambda{}_{|\nu|} \delta_{\lambda]}{}^\kappa - 4\omega_{[\nu}{}^\lambda{}_{|\mu|} \delta_{\lambda]}{}^\kappa - 12\omega_{[\mu}{}^\lambda{}_\nu \delta_{\lambda]}{}^\kappa \\
= \; & 2\left(\omega_\mu{}^\kappa{}_\nu - \omega_\nu{}^\kappa{}_\mu - \omega_\lambda{}^\lambda{}_\nu \delta_\mu{}^\kappa + \omega_\lambda{}^\lambda{}_\mu \delta_\nu{}^\kappa - \omega_\mu{}^\kappa{}_\nu - \omega_\nu{}^\lambda{}_\lambda \delta_\mu{}^\kappa - \omega_\lambda{}^\lambda{}_\mu \delta_\nu{}^\kappa + \omega_\mu{}^\lambda{}_\lambda \delta_\nu{}^\kappa + \omega_\lambda{}^\lambda{}_\nu \delta_\mu{}^\kappa + \omega_\nu{}^\kappa{}_\mu\right) \\
= \; & 2\left(-\omega_\nu{}^\lambda{}_\lambda \delta_\mu{}^\kappa + \omega_\mu{}^\lambda{}_\lambda \delta_\nu{}^\kappa\right) = 0.
\end{aligned} \tag{3.32}$$

Hence all terms involving the connection cancel, and we finally obtain

$$\left(T^\rho \wedge E^\sigma\right)\eta_{\rho\sigma\mu\nu} = 0. \tag{3.33}$$

Lect. 10

We claim that it follows from

$$T^\tau \wedge E^\lambda = 0 \tag{3.34}$$

that the torsion two-form has to vanish identically. To show this, we expand in a basis of one-forms:

$$T^\tau = \frac{1}{2} T^\tau{}_{\alpha\beta} E^\alpha \wedge E^\beta. \tag{3.35}$$

Then we have

$$T^\tau{}_{\alpha\beta} E^\alpha \wedge E^\beta \wedge E^\lambda = 0. \tag{3.36}$$

We multiply this three-form by $\eta_{\tau\lambda\rho\sigma}$ and take its components:

$$T^\tau{}_{[\alpha\beta}\eta_{|\tau|\lambda]\rho\sigma} = 0. \tag{3.37}$$

More explicitly,

$$T^\tau{}_{\alpha\beta}\eta_{\tau\lambda\rho\sigma} + T^\tau{}_{\lambda\alpha}\eta_{\tau\beta\rho\sigma} + T^\tau{}_{\beta\lambda}\eta_{\tau\alpha\rho\sigma} = 0. \tag{3.38}$$

Now we contract with $\eta^{\lambda\rho\sigma\kappa}$

$$\begin{aligned} 0 &= T^{\tau}{}_{\alpha\beta}\eta_{\tau\lambda\rho\sigma}\eta^{\lambda\rho\sigma\kappa} + T^{\tau}{}_{\lambda\alpha}\eta_{\tau\beta\rho\sigma}\eta^{\lambda\rho\sigma\kappa} + T^{\tau}{}_{\beta\lambda}\eta_{\tau\alpha\rho\sigma}\eta^{\lambda\rho\sigma\kappa} \\ &= 6T^{\tau}{}_{\alpha\beta}\delta_{\tau}{}^{\kappa} - 2\left(\delta_{\tau}{}^{\lambda}\delta_{\beta}{}^{\kappa} - \delta_{\beta}{}^{\lambda}\delta_{\tau}{}^{\kappa}\right)T^{\tau}{}_{\lambda\alpha} - 2\left(\delta_{\tau}{}^{\lambda}\delta_{\alpha}{}^{\kappa} - \delta_{\alpha}{}^{\lambda}\delta_{\tau}{}^{\kappa}\right)T^{\tau}{}_{\beta\lambda} \\ &= 6T^{\kappa}{}_{\alpha\beta} - 2\left(T^{\tau}{}_{\tau\alpha}\delta_{\beta}{}^{\kappa} - T^{\kappa}{}_{\beta\alpha}\right) - 2\left(T^{\tau}{}_{\beta\tau}\delta_{\alpha}{}^{\kappa} - T^{\kappa}{}_{\beta\alpha}\right) \\ &= 2T^{\kappa}{}_{\alpha\beta} - 2T^{\tau}{}_{\tau\alpha}\delta_{\beta}{}^{\kappa} - 2T^{\tau}{}_{\beta\tau}\delta_{\alpha}{}^{\kappa} \end{aligned} \tag{3.39}$$

Contract κ in this equation with β to find

$$2T^{\kappa}{}_{\alpha\kappa} + 8T^{\tau}{}_{\alpha\tau} - 2T^{\tau}{}_{\alpha\tau} = 0. \tag{3.40}$$

Therefore we have $T^{\kappa}{}_{\alpha\kappa} = 0$ and consequently

$$T^{\kappa}{}_{\alpha\beta} = 0. \tag{3.41}$$

We have discovered that the action

$$I[E,\omega] = \int R^{\mu\nu}(\omega) \wedge E^{\rho} \wedge E^{\sigma}\eta_{\mu\nu\rho\sigma}, \tag{3.42}$$

when E and ω are independently varied, gives implicitly a metric conection and explicitly vanishing torsion and the vacuum Einstein equations. In this sense, the action is superior to the Einstein-Hilbert action.

3.2 Yang-Mills Action

Yang-Mills theory in four dimensions can be defined by the action

$$I = \frac{1}{2}\int \mathrm{Tr}(F \wedge *F), \tag{3.43}$$

where the gauge group G is compact. (If you do not make this assumption, the quantum theory will violate unitarity.) For a set of generators $\{T_\alpha\}$ of G,

$$I = \frac{1}{2}\int \mathrm{Tr}(F^{\alpha}T_{\alpha} \wedge *F^{\beta}T_{\beta}), \tag{3.44}$$

and we can use the Cartan metric $\mathrm{Tr}(T_\alpha T_\beta) = -\frac{1}{2}\eta_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}$ (since G is compact) to rewrite this as

$$I = \frac{1}{2}\int F^{\alpha} \wedge *F^{\beta}\,\mathrm{Tr}(T_\alpha T_\beta) = \frac{1}{4}\int F^{\alpha}(A) \wedge *F_{\alpha}(A), \tag{3.45}$$

where explicitly

$$F^{\alpha} = dA^{\alpha} + \frac{1}{2}gc_{\beta\gamma}{}^{\alpha}A^{\beta} \wedge A^{\gamma}. \tag{3.46}$$

We take G to be compact and semi-simple, so that c is totally antisymmetric in all indices. If you think about it, the components of $G \wedge *F$, where G and F are two-forms, are proportional to

$$G_{\mu\nu}F^{\rho\sigma}\eta_{\rho\sigma\lambda\tau}, \tag{3.47}$$

and so the thing that is integrated is

$$G_{\mu\nu}F^{\rho\sigma}\eta_{\rho\sigma\lambda\tau}\eta^{\mu\nu\lambda\tau}. \tag{3.48}$$

But this is the same as you get for $*G \wedge F$. Hence, under variation of A one gets two identical terms:

$$\begin{aligned}
\delta I &= \frac{1}{4}\int \delta F^{\alpha}(A) \wedge *F_{\alpha}(A) + F^{\alpha}(A) \wedge *\delta F_{\alpha}(A) \\
&= \frac{1}{2}\int \delta F^{\alpha}(A) \wedge *F_{\alpha}(A) \\
&= \frac{1}{2}\int \delta\left(dA^{\alpha} + \frac{1}{2}gc_{\beta\gamma}{}^{\alpha}A^{\beta} \wedge A^{\gamma}\right) \wedge *F_{\alpha}(A) \\
&= \frac{1}{2}\int \left(\delta dA^{\alpha} + \frac{1}{2}gc_{\beta\gamma}{}^{\alpha}\delta A^{\beta} \wedge A^{\gamma} + \frac{1}{2}gc_{\beta\gamma}{}^{\alpha}A^{\beta} \wedge \delta A^{\gamma}\right) \wedge *F_{\alpha}(A) \\
&= \frac{1}{2}\int \left(\delta dA^{\alpha} + gc_{\beta\gamma}{}^{\alpha}\delta A^{\beta} \wedge A^{\gamma}\right) \wedge *F_{\alpha}(A)
\end{aligned} \tag{3.49}$$

Again, we use

$$\int_{\partial M} \delta A \wedge *F = \int_{M} d(\delta A \wedge *F) = \int_{M} (d\delta A \wedge *F - \delta A \wedge d*F) \tag{3.50}$$

and set the boundary term to zero to obtain

$$\delta I = \frac{1}{2}\int \delta A^{\alpha} \wedge \left(d*F_{\alpha} + gc_{\alpha\beta}{}^{\gamma}A^{\beta} \wedge *F_{\gamma}\right) \tag{3.51}$$

which should vanish for all δA^{α}. One obtains the **Yang-Mills equations**

$$d*F_{\alpha} + gc_{\alpha\beta}{}^{\gamma}A^{\beta} \wedge *F_{\gamma} = 0, \tag{3.52}$$

or alternatively

$$d*F + g[A, *F] = 0. \tag{3.53}$$

Conventionally one rescales the fields to remove g from the definition of F:

$$A \to \frac{A}{g}, \quad F = dA + gA \wedge A \to \frac{1}{g}(dA + A \wedge A). \tag{3.54}$$

Since the action is homogeneous of degree two in F, g can be taken outside the integral:

$$I = \frac{1}{4g^2}\int \mathrm{Tr}(F \wedge *F), \quad F = dA + A \wedge A. \tag{3.55}$$

Notice that this is just a way of rescaling the fields which is not a change of physics.

4 Topologically Non-Trivial Field Configurations

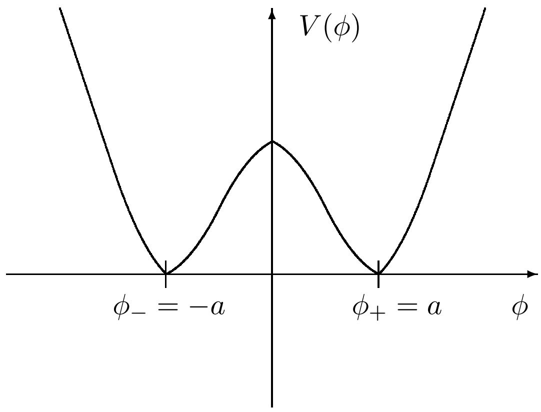
These are things which you can not see in perturbation theory in quantum field theory, but are nevertheless important. The simplest example is a domain wall in scalar field theory, where we take the potential to be

$$V(\phi) = \lambda(\phi^2 - a^2)^2. \tag{4.1}$$

This is a renormalizable sensible field theory. The action for this theory in four-dimensional spacetime is

$$I = \int d^4x\,\sqrt{g}\,\left(-\frac{1}{2}\partial_a\phi\partial_b\phi g^{ab} - V(\phi)\right), \tag{4.2}$$

where we take spacetime to be Minkowski space. The potential has minima at $\phi = \pm a$:



There are two different choices of vacuum state.
Perturbation theory describes small fluctuations around one of the minima. We might be interested, instead of doing this, in **domain walls**. These are configurations where the field takes two different asymptotic values in different regions of space.
Fluctuations around a minimum can be described as having some kind of mass. For an ordinary massive particle,

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{4.3}$$

Hence we can define

$$m^2 = V''(\text{vacuum}). \tag{4.4}$$

This describes the curvature at the minimum. For our $V(\phi)$,

$$V(\phi) = \lambda(\phi^2 - a^2)^2, \quad V'(\phi) = 4\lambda\phi(\phi^2 - a^2), \quad V''(\phi) = 4\lambda(\phi^2 - a^2) + 8\lambda\phi^2 \tag{4.5}$$

and hence

$$m^2 = V''(\pm a) = 8\lambda a^2. \tag{4.6}$$

The potential has coupling constant λ and describes particles of mass $\sqrt{8\lambda}a$. λ and m desribe the physical variables in this problem. The equations of motion (the analogue of the Klein-Gordon equation) are

$$\Box\phi - V'(\phi) = 0. \tag{4.7}$$

We look for solutions that are static and have planar symmetry. This turns it into a one-dimensional problem: If $\phi = \phi(z)$, the Klein-Gordon equation becomes

$$\frac{d^2\phi}{dz^2} = 4\lambda\phi(\phi^2 - a^2). \tag{4.8}$$

Multiply this equation by $\phi'(z)$ and integrate over z:

$$\begin{aligned}\phi''(z)\phi'(z) &= 4\lambda\phi(z)\phi'(z)(\phi(z)^2-a^2)\\ \frac{1}{2}(\phi'(z))^2 &= \lambda(\phi(z)^2-a^2)^2+\text{constant}.\end{aligned}\tag{4.9}$$

A long way away from the domain wall, we assume that $\phi\to\phi_\pm$ and $\phi'(z)\to 0$. This means the constant is set to zero. Now integrate again:

$$\begin{aligned}\phi'(z) &= \pm\sqrt{2\lambda}(\phi(z)^2-a^2)\\ \int\frac{d\phi}{\phi^2-a^2} &= \pm\sqrt{2\lambda}\int dz\\ \frac{1}{a}\text{Artanh}\frac{\phi}{a} &= \pm\sqrt{2\lambda}(z-z_0).\end{aligned}\tag{4.10}$$

Choosing the positive sign, we have found a **kink** solution

$$\phi(z)=a\tanh\left(\sqrt{2\lambda}a(z-z_0)\right).\tag{4.11}$$

The solution interpolates between two vacua. $z=z_0$ is a domain wall which separates one vacuum from a different vacuum. Lect. 11

The energy of the field is in the region around $z=z_0$. To calculate the energy-momentum tensor, take the covariant Lagrangian

$$I=\int d^4x\,\sqrt{g}\left(-\frac{1}{2}\partial_a\phi\partial_b\phi g^{ab}-V(\phi)\right),\tag{4.12}$$

then the energy-momentum tensor is given by

$$T_{ab}=-\frac{2}{\sqrt{g}}\frac{\delta I}{\delta g^{ab}}.\tag{4.13}$$

This gives

$$T_{ab}=\partial_a\phi\partial_b\phi-\frac{1}{2}g_{ab}\left(g^{cd}\partial_c\phi\partial_d\phi+V(\phi)\right)\tag{4.14}$$

You want to calculate the energy per unit area in the domain wall, which is

$$\int_{-\infty}^{\infty}dz\;T_{00}.\tag{4.15}$$

We put $z_0=0$ for simplicity. Then

$$\phi(z)=a\tanh(\sqrt{2\lambda}az),\quad \phi'(z)=\frac{\sqrt{2\lambda}a^2}{\cosh^2(\sqrt{2\lambda}az)},\tag{4.16}$$

$$\begin{aligned}T_{00} &= \frac{1}{2}\left(\phi'^2+\lambda(\phi^2-a^2)^2\right)\\ &= \frac{1}{2}\left(\frac{2\lambda a^4}{\cosh^4(\sqrt{2\lambda}az)}+\lambda a^4(\tanh^2(\sqrt{2\lambda}az)-1)^2\right)\\ &= \frac{1}{2}\lambda a^4\left(\frac{2}{\cosh^4(\sqrt{2\lambda}az)}+\frac{1}{\cosh^4(\sqrt{2\lambda}az)}\right)\\ &= \frac{3\lambda a^4}{2\cosh^4(\sqrt{2\lambda}az)}.\end{aligned}\tag{4.17}$$

The energy per unit area is then

$$\begin{aligned}
\int_{-\infty}^{\infty} dz\, T_{00} &= \frac{3\lambda a^4}{2}\int_{-\infty}^{\infty} dz\, \frac{1}{\cosh^4(\sqrt{2\lambda}az)} \\
&= \frac{3\lambda a^4}{2}\int_{-\infty}^{\infty} dz \left(\frac{1}{\cosh^2(\sqrt{2\lambda}az)} - \frac{\tanh^2(\sqrt{2\lambda}az)}{\cosh^2(\sqrt{2\lambda}az)}\right) \\
&= \frac{3\lambda a^4}{2}\left[\frac{1}{\sqrt{2\lambda}a}\tanh(\sqrt{2\lambda}az) - \frac{1}{3\sqrt{2\lambda}a}\tanh^3(\sqrt{2\lambda}az)\right]_{-\infty}^{\infty} \\
&= \frac{3\lambda a^4}{2}\frac{1}{\sqrt{2\lambda}a}\frac{4}{3} = \sqrt{2\lambda}a^3 = \frac{1}{16}\frac{m^3}{\lambda},
\end{aligned} \tag{4.18}$$

where we used $a = m/\sqrt{8\lambda}$. The important result is that this is proportional to λ^{-1}.
For static configurations, the action is energy times a time interval. The path integral will be

$$Z \sim \int D[\phi]e^{-iI[\phi]}. \tag{4.19}$$

The amplitude with any process that contains a domain wall will be

$$Z \sim e^{i/\lambda}, \tag{4.20}$$

up to numerical factors. This has an essential singularity at $\lambda = 0$.
The key point is that you can never find this process in perturbation theory in λ. Therefore one has to do things which are inherently non-perturbative in nature. The amplitudes involving topologically non-trivial configurations always involve inverse powers of the coupling constants.

5 Kaluza-Klein Theory

This, in its simplest form, is just general relativity in five dimensions instead of four. Your first reaction will be that this makes absolutely no sense.
Imagine that one dimension out of the five is wrapped up in the form of a very small circle. We would like x^5 to be wrapped up with radius R, so we identify

$$x^5 \text{ with } x^5 + 2\pi R. \tag{5.1}$$

You could argue that this is a special class of solutions which are irrelevant. But this is nothing unusual, this is simply what one does to make life easy - compare with isotropic and homogeneous solutions in cosmology. We assume there is a Killing vector associated with translations in x^5,

$$\frac{\partial}{\partial x^5} = K^a\frac{\partial}{\partial x^a}. \tag{5.2}$$

Then, the five-metric can be written as not to depend explicitly on x^5. We write this as

$$g_{ab} = \left(\begin{array}{c|c} g_{55} & g_{5j} \\ \hline g_{i5} & g_{ij} \end{array}\right), \tag{5.3}$$

where the indices i, j run from 0 to 3.
From a four-dimensional point of view, g_{5j} looks like a vector field, g_{55} looks like a scalar field. That is indicative of what you should expect.
We can write the five-metric in the following form, which is cunningly chosen to make life easy:

$$ds^2 = e^{2\beta\phi}\gamma_{ij}dx^i dx^j + e^{2\alpha\phi}\left(dx^5 + A_i dx^i\right)^2, \tag{5.4}$$

where we interpret γ_{ij} as a four-dimensional metric, A_i as a vector field under four-dimensional co-ordinate transformations and ϕ as a scalar field under four-dimensional co-ordinate transformations. Now draw your attention to what happens under an infinitesimal co-ordinate transformation that does involve x^5. Suppose that

$$A_i \to A_i + \partial_i \Lambda, \tag{5.5}$$

then the one-form $dx^5 + A_i dx^i$ is invariant if

$$x^5 \to x^5 - \Lambda. \tag{5.6}$$

If this were to describe electromagnetism, a gauge transformation is the same as a co-ordinate transformation. The simplest thing to do is to calculate the Ricci scalar, because this is what appears in the action

$$I = \frac{1}{16\pi G_N^{(5)}}\int d^5x\,\sqrt{g}\;{}^{(5)}R = \frac{1}{16\pi G_N^{(5)}}\int d^4x \underbrace{\int dx^5}_{=2\pi R}\; e^{(4\beta+\alpha)\phi}\sqrt{\gamma}\;{}^{(5)}R. \tag{5.7}$$

(Note that $\det g = e^{(8\beta+2\alpha)\phi}\det\gamma$.)
The calculation of ${}^{(5)}R$ is half messy and half straightforward. You should do the straightforward part yourself. The messy part is an application of the technology of forms.
Step 1 Find an orthonormal basis in $d = 5$: Define

$$E^5 = e^{\alpha\phi}(dx^5 + A), \quad A \equiv A_i dx^i, \tag{5.8}$$

and regard $ds^4_{(2)} = \gamma_{ij}dx^i dx^j$ as four-dimensional line element which defines an orthonormal basis of one-forms e^i, such that

$$\gamma_{ij}dx^i dx^j = e^i \otimes e^j \eta_{ij}, \tag{5.9}$$

where we now use spacetime and tangent space indices interchangably. Then the five-dimensional one-forms are defined by

$$E^i = e^{\beta\phi}e^i, \tag{5.10}$$

and the five-metric is

$$ds^2 = E^5 \otimes E^5 + \eta_{ij}E^i \otimes E^j. \tag{5.11}$$

Step 2 Calculate the connection one-forms, setting torsion to zero:

$$\begin{aligned} dE^i &= \beta\, d\phi\, e^{\beta\phi} \wedge e^i + e^{\beta\phi}de^i \\ &= \beta\, d\phi\, e^{\beta\phi} \wedge e^i - e^{\beta\phi}\hat{\omega}^i{}_j \wedge e^j, \end{aligned} \tag{5.12}$$

where we use Cartan's first equation of structure $de^i = -\hat{\omega}^i{}_j \wedge e^j$ for the four-dimensional connection $\hat{\omega}$, and

$$dE^5 = \alpha\, d\phi \wedge e^{\alpha\phi}(dx^5 + A) + e^{\alpha\phi} dA. \tag{5.13}$$

Thinking of electromagnetism, we write dA as F, with

$$F = \frac{1}{2} F_{ij} e^i \wedge e^j = \frac{1}{2}\left(\partial_i A_j - \partial_j A_i\right) e^i \wedge e^j \tag{5.14}$$

We can rewrite this in terms of the big E's with extreme ease:

$$\begin{aligned} dE^i &= \beta\, e^{-\beta\phi} \partial_j \phi\, E^j \wedge E^i - \hat{\omega}^i{}_j \wedge E^j, & (5.15)\\ dE^5 &= \alpha\, e^{-\beta\phi} \partial_j \phi\, E^j \wedge E^5 + \frac{1}{2} e^{(\alpha-2\beta)\phi}\left(F_{ij} E^i \wedge E^j\right), & (5.16) \end{aligned}$$

where $\partial_j\phi$ relates to the components of $d\phi = (\partial_j\phi)\, e^j$ (not written in terms of E^j)!
From Cartan's first equation of structure, we obtain the connection components (note that the connection must be antisymmetric)

$$\begin{aligned} \omega^5{}_i &= \alpha \partial_i \phi\, e^{-\beta\phi} E^5 + \frac{1}{2} e^{(\alpha-2\beta)\phi} F_{ij} E^j, & (5.17)\\ \omega^i{}_j &= \hat{\omega}^i{}_j - \beta e^{-\beta\phi}\left(\partial^i \phi E_j - \partial_j \phi E^i\right) - \frac{1}{2} e^{(\alpha-2\beta)\phi} F^i{}_j E^5, & (5.18) \end{aligned}$$

where we get the first from $dE^5 = -\omega^5{}_i \wedge E^i$, and the second from

$$dE^i = -\omega^i{}_j \wedge E^j - \omega^i{}_5 \wedge E^5 = -\omega^i{}_j \wedge E^j + \frac{1}{2} e^{(\alpha-2\beta)\phi} F^i{}_j E^j \wedge E^5. \tag{5.19}$$

Lect.

In slightly more general terms, we could do a reduction from $(d+1)$ dimensions to d dimensions, 12
where the metric is written as

$$ds^2 = e^{2\beta\phi} \gamma_{ij} dx^i dx^j + e^{2\alpha\phi}\left(dz + A_i dx^i\right)^2 \tag{5.20}$$

and the z direction is taken to be curled up. Of course, the calculations go through as before.
You can now calculate the two-form from this. This is rather messy and there are lots of terms you will get. We will not write the calculation out explicitly.
You will find that the result is

$$\begin{aligned} R^z{}_i = {} & e^{-2\beta\phi}\left(\alpha(\alpha - 2\beta)\partial_i\phi\partial_j\phi + \alpha\partial_j\partial_i\phi + \eta_{ij}\alpha\beta\partial_k\phi\partial^k\phi - \frac{1}{4} e^{2(\alpha-\beta)\phi} F_{kj} F^k{}_i\right) E^j \wedge E^z \\ & + e^{(\alpha-3\beta)\phi}\left(\frac{1}{2}(\alpha-\beta)\partial_i\phi F_{kj} - \frac{1}{2}(\alpha-\beta)\partial_{[k}\phi F_{j]i} - \frac{1}{2}\partial_{[k}F_{j]i} + \frac{1}{2}\beta\eta_{i[j}F_{k]l}\partial^l\phi\right) E^k \wedge E^j \\ & - \frac{1}{2} e^{(\alpha-2\beta)\phi}\left(F_{ij}\hat{\omega}^j{}_k \wedge E^k + F_{jk}\hat{\omega}^j{}_i \wedge E^k\right) + \alpha\partial_j\phi e^{-\beta\phi} E^z \wedge \hat{\omega}^j{}_i, \end{aligned} \tag{5.21}$$

$$\begin{aligned} R^i{}_j = {} & r^i{}_j + E^z \wedge E^k e^{(\alpha-3\beta)\phi}\Bigg((\alpha-\beta)F^i{}_j\partial_k\phi + \frac{1}{2}\partial_k F^i{}_j - \frac{1}{2}(\alpha-\beta)\left(\partial^i\phi F_{jk} - F^i{}_k\partial_j\phi\right) \\ & + \frac{1}{2}\beta\left(\partial_n\phi F^n{}_j\delta^i{}_k + F^i{}_l\partial^l\phi\eta_{jk}\right)\Bigg) + E^k \wedge E^l e^{-2\beta\phi}\Big(\beta\left(\partial_j\partial_{[k}\phi\delta^i{}_{l]} - \partial_{[k}\partial^i\phi\eta_{l]j}\right) \\ & - \frac{1}{4} e^{2(\alpha-\beta)\phi}\left(F^i{}_j F_{kl} + F^i{}_{[k|}F_{j|l]}\right) + \beta^2\left(\partial^i\phi\partial_{[k}\phi\eta_{l]j} - \partial_p\phi\partial^p\phi\delta^i{}_{[k}\eta_{l]j} - \partial_j\phi\partial_{[k}\phi\delta^i{}_{l]}\right)\Big) \\ & + \beta e^{-\beta\phi}\left(\partial_k\phi E^i \wedge \hat{\omega}^k{}_j - \partial^k\phi\hat{\omega}^i{}_k \wedge E_j\right) - \frac{1}{2}\beta e^{(\alpha-2\beta)\phi}\left(F^k{}_j\hat{\omega}^i{}_k \wedge E^z - F^i{}_k\hat{\omega}^k{}_j \wedge E^z\right), \end{aligned} \tag{5.22}$$

where $r^i{}_j = d\hat{\omega}^i{}_j + \hat{\omega}^i{}_k \wedge \hat{\omega}^k{}_j$ is the d-dimensional Riemann tensor.
This, as you can gather, is an unpleasant calculation to do. If we look at the action

$$I = \frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{g} R, \tag{5.23}$$

we will find that this can be brought in to an extremely simple form. From the expressions for $R^i{}_j$, discover that

$$\begin{aligned}
R &= 2R^z{}_{iz}{}^i + R^i{}_{ji}{}^j \\
&= 2e^{-2\beta\phi}\left(\alpha(-\alpha+2\beta)(\nabla\phi)^2 - \alpha\Box\phi - d\cdot\alpha\beta(\nabla\phi)^2 + \frac{1}{4}e^{2(\alpha-\beta)\phi}F^2\right) + 2\alpha\partial_k\phi e^{-\beta\phi}\hat{\omega}^{ik}{}_i \\
&\quad + e^{-2\beta\phi}r + 2e^{-2\beta\phi} + 2e^{-2\beta\phi}\Big(\beta(-d+1)\Box\phi - \frac{1}{4}e^{2(\alpha-\beta)\phi}\frac{1}{2}F^2 + \beta^2((\nabla\phi)^2(d-1) \\
&\quad - \frac{1}{2}(\nabla\phi)^2(d-1)d\Big) + 2\beta e^{-\beta\phi}(d-1)\partial_n\phi\hat{\omega}^{jn}{}_j \\
&= e^{-2\beta\phi}r - 2e^{-2\beta\phi}(\nabla\phi)^2\left(\alpha^2 + (d-2)\alpha\beta + \frac{1}{2}(d-2)(d-1)\beta^2\right) - 2e^{-2\beta\phi}\Box\phi(\alpha+\beta(d-1)) \\
&\quad + \frac{1}{4}e^{(2\alpha-4\beta)\phi}F^2 + 2e^{-\beta\phi}\partial_k\phi\hat{\omega}^{ik}{}_i(\alpha+\beta(d-1)).
\end{aligned} \tag{5.24}$$

The offending term involving the connection components can be set to zero by choosing

$$\alpha = -\beta(d-1). \tag{5.25}$$

Then

$$R = e^{-2\beta\phi}r - e^{-2\beta\phi}(d-1)(d+2)\beta^2(\nabla\phi)^2 + \frac{1}{4}e^{(2\alpha-4\beta)\phi}F^2. \tag{5.26}$$

If we recall that $\det g = e^{2(\alpha+d\beta)\phi}\det\gamma$, we obtain

$$\sqrt{g}\,R = \sqrt{\gamma}e^{-\beta\phi}\left(r(\gamma) + \frac{1}{4}e^{-\beta(2d+1)\phi}F^2 - (\nabla\phi)^2\right). \tag{5.27}$$

The result could also be (choosing $\alpha = -\beta(d-2)$, but then where are the connection terms?)

$$\sqrt{g}\,R = \sqrt{\gamma}\left(r(\gamma) - \frac{1}{4}e^{-2\beta(d-1)\phi}F^2 - \frac{1}{2}(\nabla\phi)^2\right). \tag{5.28}$$

You can do the integral over z in the action which just gives a constant $2\pi R$, and obtain the d-dimensional action

$$I = \underbrace{\frac{2\pi R}{16\pi G_N^{(d+1)}}}_{=:(16\pi G_N^{(d)})^{-1}} \int d^dx \sqrt{\gamma}e^{-\beta\phi}\left(r(\gamma) + \frac{1}{4}e^{-\beta(2d+1)\phi}F^2 - (\nabla\phi)^2\right). \tag{5.29}$$

In a region where ϕ is more or less constant, we just get a unified theory of a scalar field, an "electromagnetic field", and gravity. Probably, this theory would have languished in the physics of the 1920's, were it not for string theory. Since string theory only makes sense in higher dimensions, you have to do the same construction to get rid of the extra dimensions.

5.1 Particle Motion in Kaluza-Klein Theory

Now the first thing is to ask yourself about the motion of particles in this spacetime:

(a) Classical particle motion
This is given by geodesics, obtained by extremising the action

$$I = \int ds\; g_{ab}\dot{x}^a\dot{x}^b. \tag{5.30}$$

Let us decompose this

$$I = \int ds \left(e^{2\beta\phi}\gamma_{ij} + e^{2\alpha\phi}A_iA_j\right)\dot{x}^i\dot{x}^j + 2e^{2\alpha\phi}A_i\dot{x}^i\dot{z} + e^{2\alpha\phi}\dot{z}^2. \tag{5.31}$$

Since $\frac{\partial}{\partial z}$ is Killing,

$$\frac{\partial L}{\partial \dot{z}} = 2e^{2\alpha\phi}(\dot{z} + A_i\dot{x}^i) \tag{5.32}$$

is a constant of the motion along geodesics. (This is of course general: If K^a is Killing, and $u^a = \dot{x}^a$, then $u^b\nabla_b(K^a u_a) = 0$.)
We only want to consider $\phi =$ constant. This is because the vacuum solution will be of the form

$$\eta_{ij}dx^i dx^j + dz^2 + 2dz\, A_i dx^i + \dots \tag{5.33}$$

The last term in the one-particle action looks like a mass term. In a region where ϕ is constant, there is also a term

$$\underbrace{2e^{2\alpha\langle\phi\rangle}\dot{z}}_{=:\,q} A_i\dot{x}^i, \tag{5.34}$$

where q looks like the charge of a test particle.
Thus the motion in the z-direction corresponds to electric charge. This is why this does not make sense as a theory of electromagnetism; test particles have masses proportional to their charge. As a unified theory of gravity and electromagnetism, this theory was out of fashion until approximately 1982.

(b) Quantum-mechanical particle motion
Consider the Klein-Gordon equation

$$\left(-\Box + \frac{m^2}{\hbar^2}\right)\phi = 0. \tag{5.35}$$

$\hbar$ has been put in for a reason which will become apparent. We consider a semi-classical approximation of the form

$$\phi = Ae^{iS/\hbar}. \tag{5.36}$$

This wavefunction in the $\hbar \to 0$ limit gives you back the classical theory. Derivatives of ϕ are

$$\begin{aligned} \nabla_a\phi &= \nabla_a Ae^{iS/\hbar} + \frac{i}{\hbar}\nabla_a S Ae^{iS/\hbar}, & (5.37)\\ \Box\phi &= \Box Ae^{iS/\hbar} + 2\frac{i}{\hbar}(\nabla_a A)(\nabla^a S)e^{iS/\hbar} + \frac{i}{\hbar}\Box S\, Ae^{iS/\hbar} - \frac{1}{\hbar^2}\nabla_a S\nabla^a S\, Ae^{iS/\hbar}. & (5.38)\end{aligned}$$

Stick this into the Klein-Gordon equation to get

$$\left(-\frac{\Box A}{A} - 2\frac{i}{\hbar}\frac{\nabla_a A}{A}\nabla^a S - \frac{i}{\hbar}\Box S + \frac{1}{\hbar^2}(\nabla S)^2 + \frac{m^2}{\hbar^2}\right) = 0. \tag{5.39}$$

Multiply by $\hbar^2$ and take the limit $\hbar \to 0$ to get

$$(\nabla S)^2 + m^2 = 0. \tag{5.40}$$

So we can identify ∇S with the momentum, $u_a \propto \nabla_a S$. The velocity vector of a particle is orthogonal to the surfaces of constant phase of the wavefunction.
This means that u_a obeys the geodesic equation:

$$u^a(\nabla_a u_b) = (\nabla^a S)\nabla_a(\nabla_b S) = (\nabla^a S)\nabla_b\nabla_a S = \frac{1}{2}\nabla_b\left((\nabla S)^2\right) = \frac{1}{2}\nabla_b(-m^2) = 0. \tag{5.41}$$

The geodesic equation is absolutely inevitable quantum-mechanically.

Let us think of a Kaluza-Klein spacetime with metric Lect. 13

$$-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (dx^5)^2, \tag{5.42}$$

where the x^5 direction is curled up into a circle, and try to solve the Klein-Gordon equation for this metric. We will have

$$\left(-\Box_{(4)} - \frac{\partial^2}{\partial x^{5\,2}} + \frac{m^2}{\hbar^2}\right)\phi = 0. \tag{5.43}$$

If we separate variables

$$\phi = X(x^5)f(x_1, \ldots, x^4), \tag{5.44}$$

you will discover by the usual argument that

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^{5\,2}} = \text{constant} = k^2, \tag{5.45}$$

which gives $X(x^5) = e^{ikx^5}$ with real k for $k^2 > 0$, and $X(x^5) = e^{\pm|k|x^5}$ for $k^2 < 0$. But the wavefunction must be single-valued, so under $x^5 \mapsto x^5 + 2\pi R$, the wavefunction must not change. This means we must have $k^2 > 0$ with

$$k = \frac{n}{R}, \tag{5.46}$$

where n is an integer. k must be quantised in units of $\frac{1}{R}$.
Recall that velocity in the x^5 direction looks like electric charge. But the component of velocity in the x^5 direction is k, so charge is **quantised**.
But now we see precisely what is bad: Go back to the Klein-Gordon equation, which becomes

$$\left(-\Box_{(4)} + \left(k^2 + \frac{m^2}{\hbar^2}\right)\right)\phi = 0. \tag{5.47}$$

We see that the effective mass is also quantised, which is not observed.

5.2 Magnetic Monopoles

There is some folklore that any theory with charge quantisation has magnetic monopoles in it. Kaluza-Klein theory in dimension five is

$$I = \frac{1}{16\pi G^{(5)}} \int d^5x \sqrt{g}\, R; \tag{5.48}$$

this has as its symmetry group five-dimensional co-ordinate transformations.
Normally one would try to find interesting five-dimensional spacetimes such as Minkowski, Schwarzschild, etc. Kaluza-Klein theory means specifying that the five-dimensional spacetime must have a Killing vector that generates a circle S^1 of radius R.

From a four-dimensional spacetime, the action becomes something like

$$I = \frac{1}{16\pi G^{(4)}} \int d^4x \sqrt{g} \left(R - \frac{1}{4} F_{ij} F^{ij} e^{-\sqrt{3}\phi} - \frac{1}{2} (\nabla\phi)^2 \right); \tag{5.49}$$

where the four-dimensional Newton's constant G is

$$G^{(4)} = \frac{G^{(5)}}{2\pi R}. \tag{5.50}$$

This is four-dimensional general relativity coupled to a $U(1)$ vector field, which is the Abelian gauge invariance found in electromagnetism.
We have broken the symmetry group from five-dimensional co-ordinate transformations (diffeomorphisms) into the group of four-dimensional co-ordinate transformations $\times U(1)$. This is an instance of symmetry breaking, rather similar to what you do in Grand Unified Theories.
The vacuum solution in five dimensions in Minkowski space $\mathbb{R}^{4,1}$

$$-dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + d\tau^2; \tag{5.51}$$

the vacuum of Kaluza-Klein theory is metrically identical:

$$-dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + dz^2, \tag{5.52}$$

where the z direction is now a circle. To get from five-dimensional general relativity to four-dimensional general relativity, you need to choose to wrap one direction up to form a circle. That choice may seem restrictive, but it is simply what you do. You should really think of this as (another) example of symmetry breaking.

In the vacuum, we have $\phi = 0$, $A = 0$, $R_{ij} = 0$. You can always find another solution of a Ricci-flat four-dimensional metric times a flat fifth dimension, such as a **magnetic monopole**:

$$ds^2 = -dt^2 + \frac{1}{V(r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + V(r)(dz + 4m(1-\cos\theta)d\phi)^2, \quad V = \frac{1}{1+\frac{4m}{r}}. \tag{5.53}$$

The spacelike part of this metric is the "Euclidean" version of a four-dimensional space that has vanishing Ricci tensor, known as Taub-NUT space, and has rather strange properties in its Lorentzian

version. The solution obviously solves the five-dimensional Einstein equations.
The co-ordinate ranges are

$$0 \leq r < \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \tag{5.54}$$

$r = 0$ corresponds to a co-ordinate singularity, and z must be identified with period $8\pi m$; the radius of the Kaluza-Klein circle is $4m$. The last term in the metric represents the Kaluza-Klein direction. Thus the vector potential of the electromagnetic field has non-vanishing expectation value,

$$A = 4m(1 - \cos\theta)d\phi. \tag{5.55}$$

This looks kind of weird, but we have

$$F = dA = 4m \sin\theta \, d\theta \wedge d\phi. \tag{5.56}$$

This is a magnetic field with $F_{\theta\phi} = 4m\sin\theta$.
You will remember that as a three vector,

$$B_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma}F^{\beta\gamma}. \tag{5.57}$$

If we look near $r \to \infty$, we have

$$F^{\theta\phi} = \frac{4m\sin\theta}{r^4\sin^2\theta}, \quad B_r = \varepsilon_{r\theta\phi}F^{\theta\phi} = \underbrace{\sqrt{g^{(3)}}}_{r^2\,\sin\theta} F^{\theta\phi} = \frac{4m}{r^2}. \tag{5.58}$$

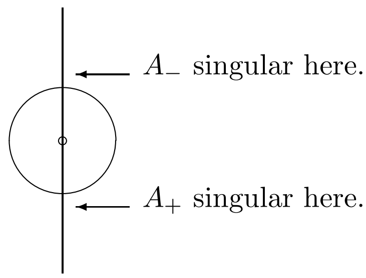
This represents a magnetic monopole of strength $4m$. All other components of the electromagnetic field fall off faster.
The vector potential is singular for $\theta = 0$, but this is a gauge artefact. We can remove this by a gauge transformation

$$A_- \mapsto A_- + d\Lambda \equiv A_+ \tag{5.59}$$

and a corresponding transformation on z. If we choose $\Lambda = -8m\phi$, then

$$A_+ - -4m(1 + \cos\theta)d\phi. \tag{5.60}$$

The magnetic field from this is the same, but the singularity has been moved to $\theta + \pi$, the south axis. Just as for co-ordinate patches in general relativity, you have found two regions which are related by a gauge transformation.



The radius of the Kaluza-Klein circle is $R = 4m = P$, the magnetic monopole strength. Electric charge for particles moving in this field is quantised in units of $\frac{1}{R}$, as we saw before. It follows that for any particle charge q, $qP = n$ must be an integer. This is the **Dirac quantisation condition**. You can discover that $r = 0$ is only a co-ordinate singularity by calculating the curvature. Near $r = 0$,

$$V(r) \sim \frac{r}{4m}. \tag{5.61}$$

The metric near $r = 0$ is

$$ds^2 = -dt^2 + \frac{4m}{r}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \frac{r}{4m}(dz + 4m(1 - \cos\theta)d\phi)^2. \tag{5.62}$$

We try to invent a co-ordinate transformation that gets rid of the singularity at $r = 0$: Under

$$\rho = \sqrt{r}, \quad dr = 2\rho\, d\rho, \tag{5.63}$$

the metric becomes

$$ds^2 = -dt^2 + 4m(4d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2) + \frac{\rho^2}{4m}(dz + 4m(1 - \cos\theta)d\phi)^2. \tag{5.64}$$

You can get rid of the constants by overall rescaling, and discover that the spatial part is the metric on flat $\mathbb{R}^4$, written as

$$d\rho^2 + \rho^2 \times (\text{metric on } S^3). \tag{5.65}$$

There is an entertaining generalization of the magnetic monopole metric, which we can write as

$$-dt^2 + \frac{1}{V(|\vec{x}|)}(d\vec{x} \cdot d\vec{x}) + V(|\vec{x}|)(dz + A), \tag{5.66}$$

where V and A satisfy

$$\nabla^2_{\text{flat}} \frac{1}{V(|\vec{x}|)} = 0 \quad (\vec{x} \neq 0), \quad \vec{\nabla} \times \vec{A} = \vec{\nabla}\frac{1}{V(|\vec{x}|)}. \tag{5.67}$$

We found that a simple pole in r in $\frac{1}{V}$ does not cause a spacetime singularity. This suggests that we can move a single monopole to $\vec{x} = \vec{x}_1$, such that

$$V = \frac{1}{1 + \frac{4m}{|\vec{x} - \vec{x}_1|}} \tag{5.68}$$

or replace it by

$$V = \frac{1}{1 + 4m\sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|}}. \tag{5.69}$$

Then $\vec{A}$ still satisfies the above relation. This is a configuration of N monopoles of strength $4m$, which are in neutral equilibrium!

5.3 S^3 as a Group Manifold

Recall that the form of the Kaluza-Klein monopole metric near $r = 0$ is Lect. 14

$$ds^2 = -dt^2 + 4m(4d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2) + \frac{\rho^2}{4m}(dz - 4m\cos\theta d\phi)^2, \tag{5.70}$$

where we have shifted z. We claimed that the spatial part of this is flat $\mathbb{R}^4$. We need to investigate the S^3 part of this metric further. Start with flat four-dimensional space with metric

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2, \tag{5.71}$$

where we define

$$x^2 + y^2 + z^2 + \tau^2 = \rho^2. \tag{5.72}$$

We want to look at the metric on surfaces of constant ρ, this will give a metric on S^3.
We use the Euler angle parametrization

$$x = \rho\cos\frac{\theta}{2}\cos\left(\frac{1}{2}(\phi - \psi)\right), \; y = \rho\cos\frac{\theta}{2}\sin\left(\frac{1}{2}(\phi - \psi)\right), \tag{5.73}$$

$$z = \rho\sin\frac{\theta}{2}\cos\left(\frac{1}{2}(\phi + \psi)\right), \; \tau = \rho\sin\frac{\theta}{2}\sin\left(\frac{1}{2}(\phi + \psi)\right). \tag{5.74}$$

The ranges of the co-ordinates are

$$0 \le \rho < \infty, \; 0 \le \theta \le \pi, \; 0 \le \phi < 2\pi, \; 0 \le \psi < 4\pi. \tag{5.75}$$

You can make life easier by assembling (x, y) and (z, τ) into a pair of complex numbers

$$u = x + iy = \rho\cos\frac{\theta}{2}\exp\left(\frac{i}{2}(\phi - \psi)\right), \; w = z + i\tau = \rho\sin\frac{\theta}{2}\exp\left(\frac{i}{2}(\phi + \psi)\right). \tag{5.76}$$

There is no escape from the following mess. To get the metric, notice that flat space in these co-ordinates has line element

$$ds^2 = du\, d\bar{u} + dw\, d\bar{w}. \tag{5.77}$$

Note that not all spaces allow for the introduction of complex co-ordinates. We have

$$du = d\rho\cos\frac{\theta}{2}e^{\frac{i}{2}(\phi-\psi)} - \frac{1}{2}\rho\sin\frac{\theta}{2}d\theta e^{\frac{i}{2}(\phi-\psi)} + \frac{i}{2}(d\phi - d\psi)\rho\cos\frac{\theta}{2}e^{\frac{i}{2}(\phi-\psi)}, \tag{5.78}$$

$$dw = d\rho\sin\frac{\theta}{2}e^{\frac{i}{2}(\phi-\psi)} + \frac{1}{2}\rho\cos\frac{\theta}{2}d\theta e^{\frac{i}{2}(\phi-\psi)} + \frac{i}{2}(d\phi + d\psi)\rho\sin\frac{\theta}{2}e^{\frac{i}{2}(\phi-\psi)}. \tag{5.79}$$

The metric of $\mathbb{R}^4$ in these co-ordinates is

$$\begin{aligned} ds^2 &= du\, d\bar{u} + dw\, d\bar{w} \\ &= d\rho^2 + \frac{1}{4}\rho^2 d\theta^2 + \frac{1}{4}\rho^2\cos^2\frac{\theta}{2}(d\phi - d\psi)^2 + \frac{1}{4}\rho^2\sin^2\frac{\theta}{2}(d\phi + d\psi)^2; \end{aligned} \tag{5.80}$$

you can see that all cross-terms cancel out. We can rewrite this metric as

$$ds^2 = d\rho^2 + \frac{1}{4}\rho^2\left(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi - \cos\theta\, d\phi)^2\right), \tag{5.81}$$

which looks like the metric we had for the magnetic monopole.
This has nothing to do with S^3 being also a **group manifold** of $SU(2)$. What do we mean by group manifold? Elements of $SU(2)$ are of the form

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad aa^* + bb^* = 1. \tag{5.82}$$

This is the same as the condition $uu^* + ww^* = 1$ (with $\rho = 1$) that we used previously.
How is one to form a metric, given that S^3 can be represented as a matrix group? We need to find a basis of one-forms and construct a metric. Suppose you have $g \in G$. The first thing to do is to construct the Lie algebra; then $g^{-1}\, dg$ will give you a basis of one-forms which are **left-invariant**: Under $g \to hg$, $g^{-1}\, dg$ is invariant.
Alternatively, one could construct right-invariant one-forms wich are invariant under $g \to gh$. Then one would use $dg\, g^{-1}$.
Suppose one starts with $g^{-1}\, dg$ and sends $g \mapsto g^{-1}$. Then

$$g^{-1}\, dg \to g\, dg^{-1}, \tag{5.83}$$

But if you take d of the equation $gg^{-1} = \mathbf{1}$, you find that

$$dg\, g^{-1} + g\, dg^{-1} = 0, \tag{5.84}$$

and hence $dg^{-1} = -g^{-1}\, dg\, g^{-1}$. So $g\, dg^{-1} = -dg\, g^{-1}$, and the inverse map maps left-invariant one-forms to right-invariant one-forms.
You can construct a **bi-invariant** metric by taking

$$-\frac{1}{2}\mathrm{Tr}(g^{-1}\, dg \otimes g^{-1}\, dg) = -\frac{1}{2}\mathrm{Tr}(dg\, g^{-1} \otimes dg\, g^{-1}) \tag{5.85}$$

where $\otimes$ is a tensor product of forms. To construct the Lie algebra of $SU(2)$, we choose generators, the **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.86}$$

They satisfy $\sigma_i \sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_K$. The Euler angle parametrization of $SU(2)$ is then

$$g = e^{i\frac{\phi}{2}\sigma_3} e^{i\frac{\theta}{2}\sigma_2} e^{-i\frac{\psi}{2}\sigma_3}. \tag{5.87}$$

These angles are almost the same co-ordinates as we used before. Use $\sigma_1^2 = \mathbf{1}$ to write this as

$$\begin{aligned} g &= \left(\mathbf{1}\cos\frac{\phi}{2} + i\sigma_3 \sin\frac{\phi}{2}\right)\left(\mathbf{1}\cos\frac{\theta}{2} + i\sigma_2 \sin\frac{\theta}{2}\right)\left(\mathbf{1}\cos\frac{\psi}{2} - i\sigma_3 \sin\frac{\psi}{2}\right) \qquad (5.88) \\ &= \left(\cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\right)\mathbf{1} + \left(-\cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\right)i\sigma_1 \\ &\quad + \left(-\sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} + \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\right)i\sigma_2 + \left(\sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\right)i\sigma_3 \\ &= \begin{pmatrix} \cos\frac{\theta}{2}e^{\frac{i}{2}(\phi-\psi)} & i\sin\frac{\theta}{2}e^{-\frac{i}{2}(\phi+\psi)} \\ i\sin\frac{\theta}{2}e^{\frac{i}{2}(\phi+\psi)} & \cos\frac{\theta}{2}e^{-\frac{i}{2}(\phi-\psi)} \end{pmatrix} \qquad (5.89) \end{aligned}$$

Clearly this is an element of $SU(2)$. You will notice that in previous notation, this is

$$g = \begin{pmatrix} u & i\bar{w} \\ iw & \bar{u} \end{pmatrix}. \tag{5.90}$$

We need to calculate $g^{-1}\,dg$:

$$g^{-1} = \begin{pmatrix} \bar{u} & -i\bar{w} \\ -iw & u \end{pmatrix}, \tag{5.91}$$

$$g^{-1}\,dg = \begin{pmatrix} \bar{u} & -i\bar{w} \\ -iw & u \end{pmatrix} \begin{pmatrix} du & i\,d\bar{w} \\ i\,dw & d\bar{u} \end{pmatrix} = \begin{pmatrix} \bar{u}du + \bar{w}dw & i\,\bar{u}\,d\bar{w} - i\,\bar{w}\,d\bar{u} \\ -i\,w\,du + i\,u\,dw & w\,d\bar{w} + u\,d\bar{u} \end{pmatrix}. \tag{5.92}$$

We use $\mathrm{Tr}\ M^2 = M_{11}^2 + 2M_{12}M_{21} + M_{22}^2$ to get the metric

$$-\frac{1}{2}\mathrm{Tr}(g^{-1}\,dg\,g^{-1}\,dg) = -\frac{1}{2}\left((\bar{u}du + \bar{w}dw)^2 + (w\,d\bar{w} + u\,d\bar{u})^2 - 2(\bar{u}\,d\bar{w} - \bar{w}\,d\bar{u})(-w\,du + u\,dw)\right). \tag{5.93}$$

To get back to the correct answer, use $u\bar{u} + w\bar{w} = 1$, and

$$du\,\bar{u} + u\,d\bar{u} + w\,d\bar{w} + dw\,\bar{w} = 0. \tag{5.94}$$

This gives

$$\begin{aligned} -\frac{1}{2}\mathrm{Tr}(g^{-1}\,dg\,g^{-1}\,dg) &= -(\bar{u}du + \bar{w}dw)^2 + (\bar{u}\,d\bar{w} - \bar{w}\,d\bar{u})(-w\,du + u\,dw) \\ &= (\bar{u}du + \bar{w}dw)(u\,d\bar{u} + w\,d\bar{w}) + (\bar{u}\,d\bar{w} - \bar{w}\,d\bar{u})(-w\,du + u\,dw) \\ &= u\bar{u}du\,d\bar{u} + w\bar{w}dw\,d\bar{w} + \bar{u}w\,d\bar{w}du + \bar{w}u\,dw\,d\bar{u} \\ &\quad -\bar{u}w\,du\,d\bar{w} + w\bar{w}\,du\,d\bar{u} + u\bar{u}\,dw\,d\bar{w} - u\bar{w}\,dw\,d\bar{u} \\ &= du\,d\bar{u} + dw\,d\bar{w}. \end{aligned} \tag{5.95}$$

In practice, this procedure does not work for any matrix bigger than 4×4.

We have a group G, and construct an element of the Lie algebra. You can think of

$$A = g^{-1}\,dg \tag{5.96}$$

as a Lie algebra valued connection one-form. Thus A can be regarded as a Yang-Mills field. A obeys the **Maurer-Cartan equations**: The first thing to calculate would be the curvature (field strength) of A. This is

$$F = dA + A \wedge A = dg^{-1} \wedge dg + A \wedge A = -g^{-1}\,dg \wedge g^{-1}\,dg + A \wedge A = 0. \tag{5.97}$$

For this reason, $g^{-1}\,dg$ is sometimes referred to as a flat connection. Such an A is usually called a pure gauge field, that means that it is just a gauge transformation of nothing. (You can see this infintesimally, if $g = 1 + \epsilon$, then $g^{-1}\,dg = d\epsilon$.)
These fields represent the classical vacuum states of Yang-Mills theory.

6 Aspects of Yang-Mills Theory

6.1 Spontaneous Symmetry Breaking

Lect. 15

We consider Yang-Mills theory with an $SU(2)$ gauge group. The Lagrangian is

$$L = -\frac{1}{4} F^{\alpha}_{ab} F^{\alpha\, ab}, \tag{6.1}$$

where

$$F^{\alpha}_{ab} = \partial_a A^{\alpha}_b - \partial_b A^{\alpha}_a + g\varepsilon^{\alpha\beta\gamma} A^{\beta}_a A^{\gamma}_b. \tag{6.2}$$

Here Greek indices run from one to three and $\varepsilon^{\alpha\beta\gamma}$ is the alternating symbol in three dimensions which gives the structure constants for $SU(2)$.
The idea of symmetry breaking is to break the gauge group G into a subgroup H by introducing a triplet of scalar fields ϕ^{α} with a potential in the Lagrangian.
We need to introduce a gauge covariant derivative because the fields ϕ^{α} are charged under $SU(2)$:

$$D_a \phi^{\alpha} = \partial_a \phi^{\alpha} + g\varepsilon^{\alpha\beta\gamma} A^{\beta}_a \phi^{\gamma}. \tag{6.3}$$

Then what you do is to add this scalar field into the action,

$$L_{\text{scalar}} = -\frac{1}{2} D_a \phi^{\alpha} D^{a} \phi^{\alpha}. \tag{6.4}$$

If that is all you have, nothing very interesting will happen; you need to add a potential. What you add is entirely and utterly up to you. We choose this such that the theory is renormalizable, which means one can have ϕ^2, ϕ^3 and ϕ^4 terms, and gauge invariant, so that V must be a singlet under $SU(2)$. This leaves two possibilities only,

$$\phi^{\alpha}\phi^{\alpha} \quad \text{and} \quad (\phi^{\alpha}\phi^{\alpha})^2, \tag{6.5}$$

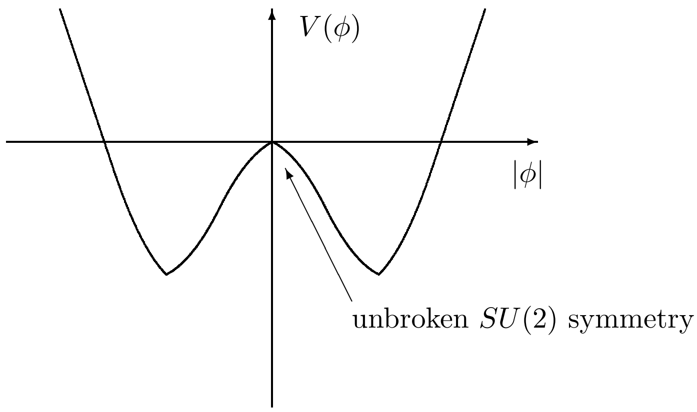
where the first is a mass term and the second a quartic coupling. For a conventional mass term,

$$V(\phi) = \frac{1}{2} m^2 \phi^{\alpha}\phi^{\alpha} + \frac{\lambda}{4} (\phi^{\alpha}\phi^{\alpha})^2, \tag{6.6}$$

where we must have $\lambda > 0$ to obtain a stable theory, the only vacuum is $\phi^{\alpha} = 0$ everywhere.
In the situation we are interested here, which corresponds to the Higgs mechanism, one changes the shape of the potential to be

$$V(\phi) = -\frac{1}{2} m^2 \phi^{\alpha}\phi^{\alpha} + \frac{\lambda}{4} (\phi^{\alpha}\phi^{\alpha})^2, \tag{6.7}$$

We obtain the familiar picture (compare with the discussion of domain walls, where one only had a single scalar field):



The phase of the theory that has unbroken $SU(2)$ is sitting at a local maximum of $V(\phi)$. This is unstable, excitations around this point have mass im and correspond to "tachyons". We are looking for vacua, these are configurations $\phi =$ constant satisfying the field equations

$$D^2\phi - V'(\phi) = 0, \tag{6.8}$$

where also $A_a \equiv 0$. We must have $V'(\phi) = 0$, hence

$$-m^2\phi^\alpha + \lambda(\phi^\alpha\phi^\alpha)\phi^\alpha = 0. \tag{6.9}$$

One possible solution is $\phi^\alpha \equiv 0$, which is called a **false vacuum**. The second solution is

$$\phi^\alpha\phi^\alpha = \frac{m^2}{\lambda}. \tag{6.10}$$

This defines a sphere in field space, which is a symmetric space $S^2 = SU(2)/U(1) = G/H$, where G is the original gauge group and H is the group which one has broken G into. It is called the **true vacuum**. The potential in the true vacuum takes the value

$$-\frac{1}{2}m^2\frac{m^2}{\lambda} + \frac{\lambda}{4}\left(\frac{m^2}{\lambda}\right)^2 = -\frac{m^4}{4\lambda} < 0. \tag{6.11}$$

This model might have cosmological implications since in the false vacuum, one would measure a vacuum energy corresponding to a "cosmological constant" relative to the true vacuum.
In the true vacuum, there is a massless excitation around the sphere which is called a **Goldstone mode**.
Now consider the gauge bosons. If you fix $\phi^1 = \phi^2 = 0$ and $\phi^3 = \frac{m^2}{\lambda}$ (say), you will discover that A^3 remains massless, while A^1 and A^2 end up with a mass term in the Lagrangian. To see the mass term, look at

$$-\frac{1}{2}D_a\phi^\alpha D^a\phi^\alpha, \quad D_a\phi^\alpha = \partial_a\phi^\alpha + g\varepsilon^{\alpha\beta\gamma}A_a^\beta\phi^\gamma. \tag{6.12}$$

There is a term

$$-\frac{1}{2}g^2\varepsilon^{\alpha\beta\gamma}A_a^\beta\phi^\gamma\varepsilon^{\alpha\delta\epsilon}A^{\delta a}\phi^\epsilon \tag{6.13}$$

in the Lagrangian. With the given values for ϕ^α, this is equal to

$$-\frac{1}{2}g^2\frac{m^4}{\lambda^2}\varepsilon^{\alpha\beta 3}A_a^\beta g\varepsilon^{\alpha\delta 3}A^{\delta a} = -\frac{1}{2}\frac{g^2m^4}{\lambda^2}\left(A_a^1A^{1a} + A_a^2A^{2a}\right), \tag{6.14}$$

which indeed is a mass term for A^1 and A^2. There is no mass term for A^3, so this massless field still has a $U(1)$ gauge symmetry.
This is a toy model for symmetry breaking, called the **Georgi-Glashow model**, which is a prototype for unifying electromagnetic and weak interactions. The masses of A^1 and A^2 are $\frac{gm^2}{\lambda}$, these correspond to the $W^\pm$ bosons; the massless A^3 corresponds to the photon.

Let us check that the number of degrees of freedom is the same in the true vaccum and in the false vacuum: Massless gauge bosons have two degrees freedom per point in space. Here we started with A^1, A^2, A^3 and ϕ^1, ϕ^2, ϕ^3, so this gives nine degrees of freedom in the false vacuum.
In the true vacuum, A^1 and A^2 are massive, and massive fields of spin s have $(2s+1)$ degrees of freedom. What you see is that A^1 and A^2 have eaten the degrees of freedom of ϕ^1 and ϕ^2.
The only field left is the scalar ϕ^3. The mass of this field at the minimum of the potential ϕ_0 is given by

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 \underbrace{V''(\phi_0)^2}_{=: m^2} + \dots \tag{6.15}$$

Here, we have

$$V''(\phi_0) = -m^2 + 3\lambda \phi^\alpha \phi^\alpha|_{\phi^\alpha \phi^\alpha = \frac{m^2}{\lambda}} = -m^2 + 3m^2 = 2m^2. \tag{6.16}$$

The mass of the Higgs boson is $\sqrt{2}m$.
This is the simplest theory in which you unify electromagnetism with something else.

6.2 Magnetic Monopoles

The action of the theory is

$$I = \int d^4x \left(-\frac{1}{2} F^\alpha_{ab} F^{\alpha\, ab} - \frac{1}{2} D_a \phi^\alpha D^a \phi^\alpha + \frac{1}{2} m^2 \phi^\alpha \phi^\alpha - \frac{\lambda}{4} (\phi^\alpha \phi^\alpha)^2 \right). \tag{6.17}$$

We found vacuum solutions by asking for ϕ and A not depending on space. The next step then is to look for static, spherically symmetric solutions.
You might think that spherical symmetry tells you that A^α_a and ϕ^α only depend on r. But this is not a gauge-invariant statement. You can only say that $\phi^\alpha \phi^\alpha$ only depends on r, since this is a singlet under the gauge group.

We are in flat space, where the metric in Cartesian co-ordinates is

$$ds^2 = -dt^2 + d\vec{x} \cdot d\vec{x}. \tag{6.18}$$

So try

$$\phi^\alpha = \frac{x^\alpha}{gr^2} H(r), \quad \phi^\alpha \phi^\alpha = \frac{H^2(r)}{g^2 r^4} (x^\alpha x^\alpha) = \frac{H^2(r)}{g^2 r^2}. \tag{6.19}$$

This is a reasonable guess as long as we do not mind using α as a spacetime index. Although this was originally a gauge group index, it can be interpreted as a spatial index here. This looks special to $SU(2)$, but you can always apply it to any $SU(2)$ subgroup of a given G.

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You then have to decide what to do with your gauge field:

$$A_0^\alpha = \frac{x^\alpha}{gr^2}J(r), \quad A_i^\alpha = \frac{\varepsilon^\alpha{}_{ij}x^j}{gr^2}(1-K(r)). \tag{6.20}$$

You find the equations of motion by substituting this ansatz into the Lagrangian and finding the Euler-Lagrange equations for H, J and K. This of course is not a mathematically correct thing to do. While it works, it is not guaranteed to work. At the end, you ought to check that H, J and K really do satisfy the equations of motion.

The equations that you get are

$$\begin{aligned} r^2K''(r) &= K(r)(K^2(r)-1) + K(r)(H^2(r)-J^2(r)), && (6.21)\\ r^2J''(r) &= 2J(r)K^2(r), && (6.22)\\ r^2H''(r) &= 2H(r)K^2(r) + \frac{\lambda}{g^2}\left(H^3(r) - \frac{m^2g^2}{\lambda^2}r^2H(r)\right). && (6.23)\end{aligned}$$

if H, J and K all go to zero as $r\to\infty$, there will be a long range classical gauge field.
You can solve these equations numerically and get a mess. What is more entertaining is that you can solve these equations analytically in a particular limit (the **Prasad-Sommerfield**[1] limit). From the Yang-Mills coupling g and the scalar field coupling λ, we can construct a length scale

$$C = \frac{mg}{\sqrt{\lambda}} \tag{6.24}$$

which controls the scale of the problem. The limit in which you can solve the equations analytically is $\lambda\to 0$ and $g\to 0$ at fixed C. The solutions are

$$K = \frac{Cr}{\sinh Cr}, \quad J=0, \quad H = Cr\coth(Cr) - 1, \tag{6.25}$$

so as $r\to\infty$, $K\to 0$ but A does not go to zero.
If one identifies the physical electromagnetic field

$$F_{ab}^{\text{(electromagn.)}} = \partial_a\left(\hat{\phi}^\alpha A_b^\alpha\right) - \partial_b\left(\hat{\phi}^\alpha A_a^\alpha\right) - \frac{1}{g}\varepsilon^{\alpha\beta\gamma}\phi^\alpha\partial_a\phi^\beta\partial_b\phi^\gamma, \tag{6.26}$$

where $\hat{\phi}^\alpha$ is a rescaled Higgs field:

$$\hat{\phi}^\alpha = \phi^\alpha\sqrt{\phi^\beta\phi^\beta}. \tag{6.27}$$

The output of this is of course well-known to all of us. As $r\to\infty$,

$$E_i = 0, \quad B_i = \frac{1}{gr^3}x_i, \tag{6.28}$$

where we define $F_{0i}^{\text{(electromagn.)}} = -E_i$ and $\frac{1}{2}\varepsilon_{ijk}F^{\text{(electromagn.)}jk} = B_i$. This is a magnetic monopole of charge $\frac{1}{g}$.

[1] Sommerfield is not a famous person!

To see this, you can define magnetic charge by the following: Take the spatial $\mathbb{R}^3$ and consider a sphere S^2 at spatial infinity. Then perform a Gaussian integral of the magnetic flux:

$$\frac{1}{4\pi}\int_{S^2_\infty} \vec{B}\cdot d\vec{S} = \frac{1}{g}. \tag{6.29}$$

Or as a differential form, you can integrate

$$\frac{1}{4\pi}\int_{S^2_\infty} F = \frac{1}{4\pi g}\int \sin\theta \, d\theta \wedge d\phi = \frac{1}{g}. \tag{6.30}$$

You should feel miserable about this for the following reason: From Stokes' theorem, you would expect

$$\frac{1}{4\pi}\int_{S^2} F = \frac{1}{4\pi}\int_{S^2} dA = \frac{1}{4\pi}\int_{\partial S^2} A = 0. \tag{6.31}$$

This argument fails because A is not globally defined. Charges of this type are regarded as **topological**. To see that A cannot be globally defined, try

$$A = \frac{1}{g}(1-\cos\theta)d\phi, \tag{6.32}$$

where the spatial metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{6.33}$$

The norm of A is

$$||A||^2 = A_0 A^0 = \frac{1}{g^2}\frac{(1-\cos\theta)^2}{r^2\sin^2\theta}. \tag{6.34}$$

Hence the norm blows up along the south axis $\theta = \pi$. To remove this, you can perform a gauge transformation which gives the same F, e.g.

$$A \to \frac{1}{g}(-1-\cos\theta)d\phi, \quad A \to A + d\Lambda, \quad \Lambda = -\frac{2}{g}\phi. \tag{6.35}$$

This then gives $||A||^2 \to \infty$ along the north axis $\theta = 0$.

6.3 Instantons in Yang-Mills Theory

The magnetic charge in the last example is of a topological nature. There are other examples which are of great interest. A second example of topological charges are **instantons**, which arise in Yang-Mills theory and gravity.

Let us consider Yang-Mills theory in flat $\mathbb{R}^4$ with positive signature, i.e. metric

$$ds^2 = {dx^1}^2 + {dx^2}^2 + {dx^3}^2 + {dx^4}^2. \tag{6.36}$$

Instantons are solutions of the Yang-Mills equations with no singularities, with finite action. We take G to be compact, then the action is

$$I = \frac{1}{4}\int F^\alpha \wedge *F_\alpha. \tag{6.37}$$

You can put F into canonical form (the normal form for an antisymmetric 4×4 matrix) by writing it as

$$F^\alpha = \frac{1}{2}F^\alpha_{ab}dx^a\wedge dx^b = F^\alpha_{12}dx^1\wedge dx^2 + F^\alpha_{34}dx^3\wedge dx^4. \tag{6.38}$$

Then

$$*F = F^\alpha_{34}dx^1\wedge dx^2 + F^\alpha_{12}dx^3\wedge dx^4. \tag{6.39}$$

The action is then

$$I = \frac{1}{2}\int d^4x\,\underbrace{(F^2_{12}+F^2_{34})}_{\geq 0}. \tag{6.40}$$

The Yang-Mills equations are

$$D_AF = 0, \quad D_A * F = 0. \tag{6.41}$$

If we are interested in solutions with finite action, we must have $F\to 0$ at infinity, and so $A = g^{-1}dg$ for some g at infinity.

It is fairly easy to discover that there is a topological charge for this system. Take the **Chern-Simons** three-form (see first example sheet)

$$CS_3 = A\wedge dA + \frac{2}{3}A\wedge A\wedge A \tag{6.42}$$

The associated topological charge is

$$\int_{S^3_\infty}\mathrm{Tr}(CS_3) = \int_{\mathbb{R}^4}\mathrm{Tr}(dCS_3) = \int_{\mathbb{R}^4}\mathrm{Tr}\left(dA\wedge dA + 2dA\wedge A\wedge A\right) \tag{6.43}$$

Recall the definition

$$F = dA + A\wedge A \tag{6.44}$$

to write this as

$$\int_{S^3_\infty}\mathrm{Tr}(CS_3) = \int_{\mathbb{R}^4}\mathrm{Tr}\left(F\wedge F - 2A\wedge A\wedge A\wedge A\right) = \int_{\mathbb{R}^4}\mathrm{Tr}\left(F\wedge F\right), \tag{6.45}$$

since the trace is invariant under cyclic permutations, which change the sign of $A\wedge A\wedge A\wedge A$. Thus

$$\frac{1}{8\pi^2}\int\mathrm{Tr}(F\wedge F), \tag{6.46}$$

called the **instanton number**, is a topological charge.

The topological charge is related in a simple way to the action: In four dimensions with Euclidean signature, $** = 1$ on two-forms. Hence $*$ has eigenvalues ±1. You can therefore decompose two-forms into **self-dual** and **anti-self-dual** parts. The self-dual part of F is

$$F_+ = \frac{1}{2}(F + *F), \tag{6.47}$$

the anti-self-dual part is

$$F_- = \frac{1}{2}(F - *F), \tag{6.48}$$

Clearly $*F_+ = F_+$ and $*F_- = -F_-$. Lect. 17

The action can be rewritten as

$$I = \frac{1}{4}\int (F_+^\alpha + F_-^\alpha)\wedge(F_+^\alpha - F_-^\alpha) = \frac{1}{4}\int \left(F_+^\alpha\wedge F_+^\alpha - F_-^\alpha\wedge F_-^\alpha\right). \tag{6.49}$$

The two remaining terms are positive definite. To see this, take F in canonical form:

$$F^\alpha = F_{12}^\alpha dx^1\wedge dx^2 + F_{34}^\alpha dx^3\wedge dx^4 \equiv F_{12}dx^{12} + F_{34}dx^{34} \tag{6.50}$$

in "cheating notation". Then

$$F_+ = \frac{1}{2}(F_{12}+F_{34})\left(dx^{12}+dx^{34}\right), \quad F_- = \frac{1}{2}(F_{12}-F_{34})\left(dx^{12}-dx^{34}\right) \tag{6.51}$$

and

$$F_+\wedge F_+ = \frac{1}{2}(F_{12}+F_{34})^2 dx^{1234}, \quad F_-\wedge F_- = -\frac{1}{2}(F_{12}-F_{34})^2 dx^{1234}. \tag{6.52}$$

Now let us compare this to the toplogical invariant we found, the "instanton number", which was

$$k = \frac{1}{8\pi^2}\int F\wedge F = \frac{1}{8\pi^2}\int (F_+ + F_-)\wedge(F_+ + F_-) = \frac{1}{8\pi^2}\int \left(F_+^\alpha\wedge F_+^\alpha + F_-^\alpha\wedge F_-^\alpha\right), \tag{6.53}$$

since $F_+\wedge F_- = 0$. Then there is a simple inequality:

$$I \geq 2\pi^2|k|, \tag{6.54}$$

with equality if and only if

$$\begin{cases} I=0, \text{ hence } F\equiv 0 & \text{for } k=0, \\ F_-\equiv 0 & \text{for } k>0, \\ F_+\equiv 0 & \text{for } k<0 \end{cases}. \tag{6.55}$$

The solutions where equality holds are self-dual, anti-self-dual, or both.
These bounds are absolutely universal in these kinds of situations in physics. The simplest instanton ought then to be self-dual. For $G = SU(2)$, this is

$$A_a(x) = \frac{1}{x^2+\lambda^2}(x_b\tau^b)^\dagger\tau_a, \tag{6.56}$$

where τ_1, τ_2 and τ_3 are Pauli matrices and $\tau^4 = \mathbf{1}$.
This self-dual solution has instanton number $k=1$, and λ defines an arbitrary scale.

Physically instantons can be interpreted as how tunnelling proceeds in the context of quantum field theory. The mathematics of instantons are a fascinating topic in their own right.

7 Gravitational Instantons

7.1 Topological Quantum Numbers for Gravity

Gravitational instantons have a number of similarities and a number of differences. We consider non-singular solutions of the Einstein equations $R_{ab} = \Lambda g_{ab}$ which have positive signature $(++++)$. The action for gravity is a bit of an embarrassment:

$$I = -\frac{1}{16\pi}\int d^4x\,\sqrt{g}\,(R-2\Lambda) \quad + \text{possible boundary terms (neglected here).} \tag{7.1}$$

The big problem here is that, unlike in Yang-Mills theory, this has no nice boundedness properties. This usually suggests an instability in the theory.
The simplest way to see the unboundedness is to do a conformal transformation

$$g_{ab} \to \hat{g}_{ab} = \Omega^2(x) g_{ab}. \tag{7.2}$$

You can do a calculation and see that

$$I[\hat{g}] = -\frac{1}{16\pi} \int d^4x \sqrt{g} \left(\Omega^2(x) R + 6(\nabla\Omega(x))^2 - 2\Lambda\,\Omega^4 \right) \tag{7.3}$$

up to integration by parts. To get this you need to find the conformally rescaled R (see example sheet).
We are not asking for solutions to the equations of motion, but think about a general variation of the action. As long as $\Lambda > 0$, the $\Lambda\,\Omega^4$ term corresponds to a positive potential, so that is fine. But the term involving $(\nabla\Omega)^2$ has the wrong sign. The action can be made arbitrarily negative by picking a rapidly oscillating Ω.

In gravity, there are two **topological quantum numbers**:

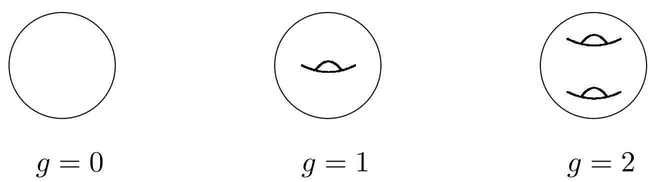
1. The **Euler character**. In two dimensions, this is

$$\chi = \frac{1}{4\pi} \int_\Sigma R. \tag{7.4}$$

If Σ is compact and orientable, χ classifies these surfaces. You can write it as

$$\chi = 2 - 2g, \tag{7.5}$$

where g is the genus of Σ. Examples are



Alternatively,

$$\chi = 2 - b_1, \tag{7.6}$$

where b_1 is the first **Betti number**. b_p is the number of S^p that cannot be contracted to a point or deformed into each other.
On S^2, you can always contract a circle to a point, so $b_1 = 0$. If you can catch the manifold with a piece of string, it has $b_1 > 0$.

There is a theorem by **Hodge** which states that *for compact manifolds, b_p is equal to the number of square integrable harmonic p-forms*, these are p-forms satisfying

$$dp = 0,\ d * p = 0, \quad \int p_a p^a < \infty. \tag{7.7}$$

It immediately follows that for these manifolds, $b_p = b_{d-p}$, which is known as **Poincaré duality**.

For $d = 4$, the Euler character can be written as

$$\chi = \frac{1}{32\pi^2} \int R^{ab} \wedge *R_{ab} = \frac{1}{128\pi^2} \int d^4x\, \sqrt{g}\, \varepsilon_{abcd} \varepsilon^{abef} R^{cdgh} R_{ghef}, \tag{7.8}$$

where you can have extra boundary terms if there is a boundary. In terms of Betti numbers,

$$\chi = 2 - 2b_1 + b_2. \tag{7.9}$$

2. The **Hirzebruch signature**

$$\tau = \frac{1}{48\pi^2} \int R^{ab} \wedge R_{ab} = \frac{1}{96\pi^2} \int d^4x\, \sqrt{g}\, R^{abcd} R_{abef} \varepsilon_{cd}{}^{ef}. \tag{7.10}$$

It, too, has a topological interpretation in terms of Betti numbers:

$$\tau = b_2^+ - b_2^-, \tag{7.11}$$

where $b_2 = b_2^+ + b_2^-$ and b_2^+ is the number of self-dual harmonic square integrable 2-forms and b_2^- is the number of anti-self-dual such forms.

There are generalizations of the Euler character for all even dimensions, and of the Hirzebruch signature for all d which are multiples of four.
These look a bit like action and instanton number for Yang-Mills theory, respectively.
The intellectual history of general relativity is littered with the corpses of people who tried to manipulate the Euler character into the action for general relativity. But general relativity just is not like Yang-Mills theory.

The inequality analogous to $I \geq 2\pi^2 |k|$ for Yang-Mills theory is

$$2\chi \geq 3|\tau|, \tag{7.12}$$

with equality if and only if the Riemann tensor is self-dual or anti-self-dual (see example sheet). Self-duality here means

$$R_{abcd} = \frac{1}{2} \varepsilon_{abef} R^{ef}{}_{cd}. \tag{7.13}$$

7.2 De Sitter Space

As an example of a gravitational instanton, we want to consider de Sitter spacetime, written in static co-ordinates

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{7.14}$$

where $\Lambda > 0$. This satisfies $R_{ab} = \Lambda g_{ab}$.
The instanton associated with de Sitter spacetime is found by sending t to $i\tau$. This preserves the field equations. The metric of the instanton is

$$ds^2 = \left(1 - \frac{\Lambda}{3}r^2\right) d\tau^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{7.15}$$

De Sitter space can be easily viewed as an hyperboloid embedded in $\mathbb{R}^{4,1}$ with metric Lect. 18

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2. \tag{7.16}$$

The hyperboloid is

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda} =: \alpha^2 > 0. \tag{7.17}$$

In FRW co-ordinates, you can view de Sitter space as a space of constant spatial curvature $k = 1$ (you could also view it as $k = 0$ or $k = -1$). Here the co-ordinates cover the entirety of de Sitter space. They are

$$\begin{gathered} v = \alpha \sinh\left(\frac{t}{\alpha}\right), \quad w = \alpha\cosh\left(\frac{t}{\alpha}\right)\cos\chi, \quad x = \alpha\cosh\left(\frac{t}{\alpha}\right)\sin\chi\cos\theta, \\ y = \alpha\cosh\left(\frac{t}{\alpha}\right)\sin\chi\sin\theta\cos\phi, \quad z = \alpha\cosh\left(\frac{t}{\alpha}\right)\sin\chi\sin\theta\sin\phi. \end{gathered} \tag{7.18}$$

If you are interested in the line element in the (t, χ, θ, ϕ) co-ordinates, you will discover it is

$$ds^2 = -dt^2 + \alpha^2\cosh^2\left(\frac{t}{\alpha}\right) \underbrace{\left(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{\text{Metric on } S^3 \text{ in hyperspherical co-ordinates}}. \tag{7.19}$$

The scale factor for this universe is

$$a(t) = \alpha\cosh\left(\frac{t}{\alpha}\right), \tag{7.20}$$

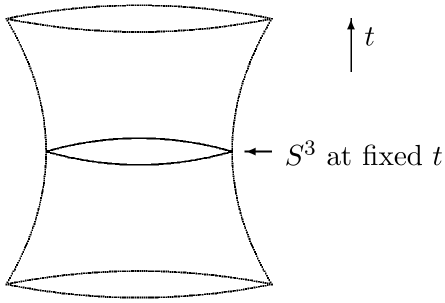
the Hubble parameter is

$$\frac{\dot{a}}{a} = \frac{1}{\alpha}\tanh\left(\frac{t}{\alpha}\right), \tag{7.21}$$

the acceleration is

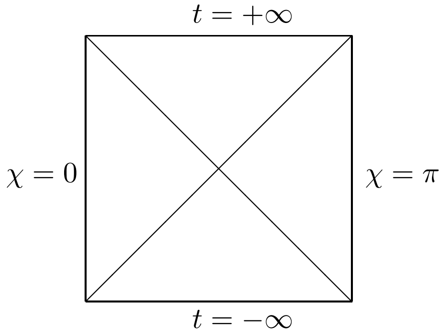
$$\frac{\ddot{a}}{a} = \frac{1}{\alpha^2} = \text{constant} > 0. \tag{7.22}$$

We can picture de Sitter space as the hyperboloid



What makes the spacetime interesting is the presence of cosmological horizons.

To the future of the light-cone of an observer moving on a timelike geodesic is a region of spacetime the observer cannot see. This is not an event horizon. These regions are different for different observers, the horizon is called a **cosmological horizon**. This leads to all kinds of interesting paradoxes.
The Penrose diagram is



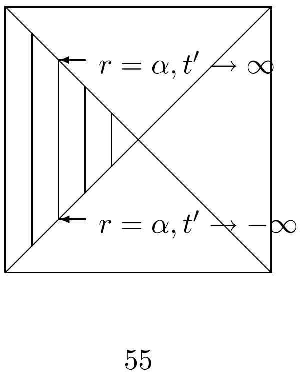
where we have drawn cosmological horizons for particular observers.
That looks a bit like the Penrose diagram for a black hole, but horizons depend on the observer.
Now we want to construct Schwarzschildesque co-ordinates for this spacetime:

$$v = \alpha\sqrt{1-\frac{r^2}{\alpha^2}}\sinh\left(\frac{t'}{\alpha}\right), \quad w = \alpha\sqrt{1-\frac{r^2}{\alpha^2}}\cosh\left(\frac{t'}{\alpha}\right),$$
$$x = r\cos\theta, \quad y = r\sin\theta\cos\phi, \quad z = r\sin\theta\sin\phi. \tag{7.23}$$

Then the line element is, as claimed above,

$$ds^2 = -\left(1-\frac{r^2}{\alpha^2}\right)dt'^2 + \frac{dr^2}{1-\frac{r^2}{\alpha^2}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{7.24}$$

which only makes sense for $0 < r < \alpha$. This only covers the following region of de Sitter space:



In cosmology, calculations are normally done in $k = 0$ co-ordinates, but these do not cover the entire spacetime. You should better use $k = 1$ co-ordinates.
Now look at the **de Sitter instanton**: Find a "Euclidean" solution to $R_{ab} = \Lambda g_{ab}$ by a co-ordinate transformation $t' \to i\tau$

$$ds^2 = \left(1 - \frac{r^2}{\alpha^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{r^2}{\alpha^2}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{7.25}$$

This is a perfect local solution, but has a singularity at $r = \alpha = \sqrt{\frac{3}{\Lambda}}$. This is actually a co-ordinate singularity. To see this, define new co-ordinates. Since we want to analyse what happens near $r = \sqrt{\frac{3}{\Lambda}}$, set

$$r = \sqrt{\frac{3}{\Lambda}}\left(1 - \frac{\Lambda}{6}\delta^2\right), \tag{7.26}$$

then

$$dr = -\sqrt{\frac{\Lambda}{3}}\delta\, d\delta, \quad r^2 \approx \frac{3}{\Lambda} - \delta^2, \tag{7.27}$$

and the line element becomes

$$ds^2 = \frac{\Lambda}{3}\delta^2\, d\tau^2 + d\delta^2 + \frac{3}{\Lambda}(d\theta^2 + \sin^2\theta\, d\phi^2) + \text{ terms higher order in } \delta. \tag{7.28}$$

The singularity is now at $\delta = 0$. This metric now consists of a metric on S^2 and a part which is the metric on a plane (almost).
The metric on a plane with co-ordinates (ρ, ψ) is

$$ds^2 = d\rho^2 + \rho^2\, d\psi^2. \tag{7.29}$$

At $\rho = 0$, you need to identify ψ with $\psi + 2\pi$. If not, you get a conical singularity; then there is a defect angle Δ and $\rho = 0$ has a δ-function in curvature. In the present case, $\delta = 0$ is a co-ordinate singularity as long as τ is identified with period $\sqrt{\frac{3}{\Lambda}}2\pi$.
This has some interesting physical consequences (see Black Holes course).

We already know that the solution is four-dimensional and has constant curvature

$$R_{abcd} \propto (g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{7.30}$$

It has ten Killing vectors, and so must be the metric on S^4. You can embed it into $\mathbb{R}^5$. We want to evaluate the topological quantum numbers.
First try to get the proportionality constant. Contract R_{abcd} to get

$$R_{ac} = 3(\text{constant})g_{ac} = \Lambda g_{ac}, \quad \text{constant} = \frac{\Lambda}{3}. \tag{7.31}$$

The Hirzebruch signature of this space is

$$\tau = \frac{1}{96\pi^2}\int d^4x\,\sqrt{g}\,\varepsilon^{ab}{}_{ef}R_{abcd}R^{cdef} = 0. \tag{7.32}$$

All even-dimensional spheres have Euler character two, all odd-dimensional spheres have $\chi = 0$. We can compute it explicitly using

$$\chi = \frac{1}{128\pi^2} \int d^4x \, \sqrt{g} \, \varepsilon_{abcd} \varepsilon_{efgh} R^{abef} R^{cdgh}. \tag{7.33}$$

Whilst this is a true formula, it is inconvenient for this calculation. You can show (on the example sheet) that an equivalent formula gives

$$\chi = \frac{1}{128\pi^2} \int d^4x \, \sqrt{g} \left(R^{abcd} R_{abcd} - 4R^{ab} R_{ab} + R^2 \right) = 2. \tag{7.34}$$

The action for the de Sitter instanton is

$$I = -\frac{1}{16\pi} \int d^4x \, \sqrt{g} \, (R - 2\Lambda) = -\frac{\Lambda}{8\pi} \cdot \text{Vol}, \tag{7.35}$$

which is negative!
From Schwarzschildesque co-ordinates, you get $\sqrt{g} = r^2 \sin\theta$. Then the volume of S^4 is

$$\text{Vol} = \int d\tau \, dr \, d\theta \, d\phi \, r^2 \sin\theta = 4\pi \int_0^{\sqrt{\frac{3}{\Lambda}}} r^2 dr \int_0^{2\pi\sqrt{\frac{3}{\Lambda}}} d\tau = \frac{8\pi^2}{3} \sqrt{\frac{3}{\lambda}} \left(\frac{3}{\Lambda} \right)^2 = \frac{24\pi^2}{\Lambda^2}, \tag{7.36}$$

and the action is

$$I = -\frac{3\pi}{\Lambda}. \tag{7.37}$$

The partition function in statistical mechanics would be

$$Z = \int D\phi \, e^{-I[\phi]}, \tag{7.38}$$

where one integrates over all metrics that are periodic in complex time, with period $\frac{1}{T}$. It follows that the temperature of de Sitter space is

$$T = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}}. \tag{7.39}$$

It is believed, but nobody has proved, that for fixed Λ this is a lower bound of the action, so that always

$$I \geq I_{\text{deSitter}}, \tag{7.40}$$

with equality only for de Sitter.

7.3 Other Examples

Now we can find other instantons, some of which are interesting, all of which are fun:

1. $S^2 \times S^2$, a direct product of two two-spheres (of radius a).
   The metric is a direct product

$$ds^2 = \begin{pmatrix} \text{Metric on } S^2 & 0 \\ 0 & \text{Metric on } S^2 \end{pmatrix}. \tag{7.41}$$

These are two Einstein manifolds, satisfying $R_{ab} = \Lambda g_{ab}$. Lect. 19

The Ricci tensor also factorises:

$$R_{ab} = \begin{pmatrix} \frac{1}{a^2} \cdot \text{Metric on } S^2 & 0 \\ 0 & \frac{1}{a^2} \cdot \text{Metric on } S^2 \end{pmatrix}. \tag{7.42}$$

Then the Ricci scalar is $\frac{4}{a^2}$, and the metric solves Einstein's equations with $\Lambda = \frac{1}{a^2}$.
The topological quantum numbers are (Exercise)

$$\chi = 4, \quad \tau = 0. \tag{7.43}$$

The action is negative:

$$I = -\frac{1}{16\pi} \int d^4x \sqrt{g}\,(R - 2\Lambda) = -\frac{\Lambda}{8\pi} \int d^4x \sqrt{g} = -\frac{\Lambda}{8\pi}(4\pi a^2)^2 = -2\pi\Lambda a^4 = -\frac{2\pi}{\Lambda}. \tag{7.44}$$

2. Fubini-Study Metric on $\mathbb{CP}^2$ (a compact manifold)
   This is

$$ds^2 = \frac{dr^2}{\left(1 + \frac{\Lambda r^2}{6}\right)^2} + \frac{1}{4} r^2 \left(\frac{1}{1 + \frac{\Lambda r^2}{6}} (d\psi \pm \cos\theta\, d\phi)^2 + d\theta^2 + \sin^2\theta\, d\phi^2 \right) \frac{1}{1 + \frac{\Lambda r^2}{6}}. \tag{7.45}$$

   If $\Lambda = 0$, then this is just flat space.
   For $\Lambda > 0$, it has finite volume and is non-singular.
   The topological quantum numbers are

$$\chi = 3, \quad \tau = 1. \tag{7.46}$$

   The action is again negative,

$$I = -\frac{9\pi}{4\Lambda}. \tag{7.47}$$

   It has interesting property of its curvature, regarding the Weyl tensor $C^a{}_{bcd}$. Consider the **Weyl two-form**

$$C^a{}_b = \frac{1}{2} C^a{}_{bcd} dx^c \wedge dx^d. \tag{7.48}$$

   Then for $\mathbb{CP}^2$, the Weyl two-form is (anti-)self-dual,

$$*C^a{}_b = \pm C^a{}_b, \tag{7.49}$$

   depending on the choice of sign in the metric.
   All of the examples presented here are of great interest in string theory.

3. **K3** is the unique simply connected compact four-manifold that satisifies $R_{ab} = 0$.
   The proof of this does not rely on the construction of a metric; the metric is not known. We know that the topological quantum numbers are

$$\chi = 24, \quad \tau = \pm 16. \tag{7.50}$$

The curvature form must be either self-dual or anti-self-dual.
There is a 58-parameter family of solutions of $R_{ab} = 0$.
To see this, consider the Betti numbers: Since K3 is simply connected, $b_1 = 0$, so

$$\chi = 2 - 2b_1 + b_2^+ + b_2^- = 2 + b_2^+ + b_2^-, \quad \tau = b_2^+ - b_2^-, \tag{7.51}$$

which tells you that

$$b_2^+ = 19, \quad b_2^- = 3. \tag{7.52}$$

There are 19 self-dual harmonic two-forms with components F^I_{ab} ($I = 1, \ldots, 19$) and three anti-self-dual harmonic two-forms with components G^J_{ab} ($J = 1, 2, 3$).
Assume that we have a metric g which solves

$$R_{ab}[g] = 0. \tag{7.53}$$

Under a slight deformation of g, if

$$R_{ab}[g + \epsilon h] = 0, \quad \epsilon \ll 1, \tag{7.54}$$

then $g + \epsilon h$ is a new solution of the Einstein equations.
The condition for this (cf. gravitational waves, GR course) is the following differential equation

$$-\Box h_{ab} - 2R_{acbd}h^{cd} = 0, \tag{7.55}$$

where h is not of the form $\nabla_{(a}\xi_{b)}$, which would just be a co-ordinate transformation. These two conditions can be summarised as

$$\nabla^a h_{ab} = 0. \tag{7.56}$$

We know that F^I_{ab} and G^J_{ab} obey

$$\nabla_a F^{Iab} = 0,\ F^{Iab} = \frac{1}{2}\varepsilon^{abcd}F^I_{cd}; \quad \nabla_a G^{Jab} = 0,\ G^{Jab} = -\frac{1}{2}\varepsilon^{abcd}G^J_{cd}. \tag{7.57}$$

You can construct

$$h_{ab} = G^J_{ac}F^{Ic}{}_b + G^J_{bc}F^{Ic}{}_a, \tag{7.58}$$

which generically will give 57 independent perturbations, one for each combination of I and J.
This is transverse, $\nabla^a h_{ab} = 0$, so not a co-ordinate transformation, and satisfies

$$-\Box h_{ab} - 2R_{acbd}h^{cd} = 0. \tag{7.59}$$

There must be one more such deformation. Since no scale is associated with this metric, it can be multiplied by a constant. If g_{ab} obeys $R_{ab} = 0$, then so does λg_{ab} for any constant λ.
The interesting thing to show is that this h_{ab} actually satisfies the wave equation (see example sheet).

If the curvature is self-dual, this tells you something about **holonomy**.

A vector at any given point p can be parallelly transported around a closed loop C. Then the components of the vector after and before parallel transport, written in some orthonormal basis, satisfy a relation

$$V^a(C,p) = M^a{}_b(C)V^b(p). \tag{7.60}$$

What are the properties of the matrix M? Parallel transport does not change the norm, so $M(C)$ is a Lorentz transformation. This is because one uses a metric-preserving connection. Therefore in the Euclidean case,

$$M \in SO(4). \tag{7.61}$$

What is remarkably true for any metric with self-dual (or anti-self-dual) curvature is that $M \in SU(2)$. (This is a consequence of supersymmetry.)

4. Another example of huge importance in string theory is the **Eguchi-Hanson metric** on a non-compact manifold. This is an attempt to construct the analogue of the Yang-Mills intanton.
   You start from the assumption that curvature should go to zero on an enormous S^3 near infinity. You start with a metric on $\mathbb{R}^4$:

$$ds^2 = \frac{1}{f(r)}dr^2 + \frac{1}{4}r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2 + f(r)(d\psi \pm \cos\theta\, d\phi)^2\right), \tag{7.62}$$

   where θ, ϕ, ψ are Euler angles on S^3 with ranges

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \psi < 4\pi \tag{7.63}$$

   which cover all of S^3, and $0 \leq r < \infty$.
   This looks very reasonable. The function f can be determined to be

$$f(r) = 1 - \left(\frac{a}{r}\right)^4, \tag{7.64}$$

   where a is an arbitrary scale. This falls off very fast as $r \to \infty$, so the metric is asymptotically flat. There is a problem at $r = a$, which is identified as a co-ordinate singularity.
   You can look at the (r,ψ) plane to discover this is a conical singularity. What is the condition that the singularity at $r = a$ is removed? We have done this many times, it tells you that the period of ψ is 2π.
   But we set the period of ψ to be 4π before. You can take the solution to be **asymptotically locally Euclidean**, such that the boundary at infinity is not S^3 but $S^3/\mathbb{Z}_2$, a three-sphere with antipodal points identified. (This is useful in string theory.)
   The topological quantum numbers of this metric are

$$\chi = 0, \quad \tau = \pm 1, \tag{7.65}$$

   where the given formulae need boundary corrections.
   The curvature is again self-dual (or anti-self-dual). The action (which also requires boundary corrections) is zero.

It can be generalised to the **Multi-Eguchi-Hanson metric**, which is of the form

$$ds^2 = \frac{1}{V(\vec{x})}(d\psi + \omega_i dx^i)^2 + V(\vec{x})\delta_{ij}dx^i\, dx^j. \tag{7.66}$$

This metric looks like the multi-monopole metric in Kaluza-Klein theory, but with $t =$ constant. Whereas in Kaluza-Klein theory we had

$$V = 1 + \lambda \sum_{i=1}^{k} \frac{1}{|\vec{x} - \vec{x}_i|}, \tag{7.67}$$

for the Multi-Eguchi-Hanson metric, we delete the one:

$$V = \lambda \sum_{i=1}^{k} \frac{1}{|\vec{x} - \vec{x}_i|}, \tag{7.68}$$

where as usual,

$$\vec{\nabla} V = \vec{\nabla} \times \vec{\omega}. \tag{7.69}$$

$k = 1$ is flat space, $k = 2$ is the Eguchi-Hanson metric; $k > 2$ is a Multi-Eguchi-Hanson metric. These have self-dual curvature, boundary $S^3/\mathbb{Z}_{k-2}$, and zero action.

8 Positive Energy

8.1 Geometry of Surfaces

Lect. 20

Consider a $(d-1)$-dimensional surface Σ, embedded in a d-dimensional manifold (possibly spacetime). If the surface is characterised by an equation $f(x) = 0$, the unit normal to the surface is

$$n_a = N\, \partial_a f, \tag{8.1}$$

where N is a normalisation factor. In a Riemannian manifold, one can always normalise n_a such that $n_a n^a = 1$.
In Lorentzian signature, you can have a timelike n_a (then Σ is called spacelike), a spacelike n_a (then Σ is called timelike), or a null n_a (then Σ is called null). We will ignore the case where $n_a n^a = 0$ since it is more difficult.
We can assume in the following that

$$n^a n_a = \pm 1. \tag{8.2}$$

We need other quantities to characterise the geometry of Σ: We can define a symmetric rank two tensor h, called the **first fundamental form** for historical reasons, by

$$h^a{}_b = \delta^a{}_b \mp n^a n_b. \tag{8.3}$$

It has the following properties.

1. h is a projection.

$$\begin{aligned} h^a{}_b h^b{}_c &= (\delta^a{}_b \mp n^a n_b)(\delta^b{}_c \mp n^b n_c) \\ &= \delta^a{}_c \mp n^a n_c \mp n^a n_c + n^a \underbrace{n_b n^b}_{\pm 1} n_c \\ &= \delta^a_c \mp n^a n_c = h^a{}_c. \end{aligned} \tag{8.4}$$

2. In d dimensions, the trace of h is

$$h^a{}_a = \delta^a{}_a \mp n^a n_a = d - 1. \tag{8.5}$$

3. h is orthogonal to n in the following sense:

$$h^a{}_b n_a = (\delta^a{}_b \mp n^a n_b) n_a = n_b \mp \underbrace{n_a n^a}_{\pm 1} n_b = 0. \tag{8.6}$$

You can deduce that any vector Y^a defined on Σ can be decomposed into two parts:

$$Y^a = \delta^a{}_b Y^b = h^a{}_b Y^b \pm n^a n_b Y^b, \tag{8.7}$$

which is a part tangential to Σ and a part perpendicular to Σ. Indeed, the first part is orthogonal to n^a, while the second is annihilated by h.
Absolutely any vector or tensor can be decomposed in this way. In particular, you can project the metric tensor into the surface Σ. You get

$$h^a{}_c h^b{}_d g_{ab} = h^a{}_c h_{ad} = h_{cd}. \tag{8.8}$$

You can consider h as the **induced metric** on Σ.
There is also a **second fundamental form**, which describes how n changes as one moves around Σ:

$$K_{cd} = h^a{}_c h^b{}_d \nabla_a n_b. \tag{8.9}$$

This is symmetric too: Since $n_a = N \nabla_a f$, we have

$$\nabla_a n_b = N \nabla_a \nabla_b f + \nabla_a N \nabla_b f = N \nabla_a \nabla_b f + \frac{\nabla_a N}{N} n_b. \tag{8.10}$$

Then

$$K_{cd} = h^a{}_c h^b{}_d \left(N \nabla_a \nabla_b f + \frac{\nabla_a N}{N} n_b \right) = h^a{}_c h^b{}_d N \nabla_a \nabla_b f, \tag{8.11}$$

since h annihilates n. This is symmetric for a torsion-free connection.
The covariant derivative of n will have components tangential to Σ and perpendicular to Σ:

$$\begin{aligned} \nabla_a n_b &= (h^c{}_a \pm n^c n_a)(h^d{}_b \pm n^d n_b) \nabla_c \nabla_d \\ &= K_{ab} + (\pm n^c n_a h^d{}_b \pm h^c{}_a n^d n_b) \nabla_c n_d + n^c n_a n^d n_b \nabla_c n_d \\ &= K_{ab} \pm n^c n_a h^d{}_b \nabla_c n_d \\ &= K_{ab} \pm n^c n_a (\delta^d{}_b \pm n_b n^d) \nabla_c n_d \\ &= K_{ab} \mp n_a \omega_b, \end{aligned} \tag{8.12}$$

where we have used

$$n^d\nabla_c n_d = \frac{1}{2}\nabla_c(n^d n_d) = 0 \tag{8.13}$$

and

$$\omega_b = -n^c\nabla_c n_b \tag{8.14}$$

is sometimes called the acceleration vector.

There is a notion of covariant derivative in Σ, which is defined by projecting the covariant derivative of the d-dimensional manifold into Σ:

$$^{(d-1)}\nabla_e T^{a\dots}{}_{b\dots} = h^{e'}{}_e h^a{}_{a'}\dots h^{b'}{}_b\dots\nabla_{e'}T^{a'\dots}{}_{b'\dots} \tag{8.15}$$

That may seem a bit perverse, but it is actually quite useful.

If you take ∇ to be the symmetric metric connection,

$$\begin{aligned}
^{(d-1)}\nabla_a^{(d-1)}\nabla_b f &= h^{a'}{}_a h^{b'}{}_b\nabla_{a'}(h^x{}_{b'}\nabla_x f)\\
&= h^{a'}{}_a h^x{}_b\nabla_{a'}\nabla_x f + h^{a'}{}_a h^{b'}{}_b(\nabla_{a'}h^x{}_{b'})(\nabla_x f)\\
&= h^{a'}{}_a h^x{}_b\nabla_{a'}\nabla_x f + h^{a'}{}_a h^{b'}{}_b(\nabla_{a'}(\delta^x{}_{b'}\mp n^x n_{b'}))(\nabla_x f)\\
&= h^{a'}{}_a h^x{}_b\nabla'_{a'}\nabla_x f \mp h^{a'}{}_a h^{b'}{}_b n^x(\nabla_{a'}n_{b'})(\nabla_x f),
\end{aligned} \tag{8.16}$$

where we used $h^{b'}{}_b n_{b'} = 0$. The first term is symmetric, the second term is proportional to K_{ab}, which we know is symmetric.

So $^{(d-1)}\nabla_a$ is a symmetric connection and its torsion vanishes. It also turns out to be a metric connection:

$$\begin{aligned}
^{(d-1)}\nabla_c h_{ab} &= h^g{}_c h^e{}_a h^f{}_b\nabla_g h_{ef}\\
&= h^g{}_c h^e{}_a h^f{}_b\nabla_g(g_{ef}\mp n_e n_f)\\
&= \mp h^g{}_c h^e{}_a h^f{}_b n_f\nabla_g n_e \mp h^g{}_c h^e{}_a h^f{}_b n_e\nabla_g n_f = 0.
\end{aligned} \tag{8.17}$$

Thus $^{(d-1)}\nabla$ is the unique symmetric metric connection of h. One can find the curvature of it by calculating

$$\left(^{(d-1)}\nabla_a^{(d-1)}\nabla_b - ^{(d-1)}\nabla_b^{(d-1)}\nabla_a\right)V_c \stackrel{!}{=} {}^{(d-1)}R_{abc}{}^d V_d \tag{8.18}$$

for a vector V_c lying in Σ, i.e. satisfying $n^c V_c = 0$. Doing this calculation requires a certain amount of concentration:

$$\begin{aligned}
^{(d-1)}R_{abc}{}^d V_d &= h^p{}_a h^q{}_b h^r{}_c\nabla_p(h^x{}_q h^y{}_r\nabla_x V_y) - (a\leftrightarrow b) \qquad (8.19)\\
&= h^p{}_a h^q{}_b h^r{}_c\nabla_p((\delta^x{}_q\delta^y{}_r \mp n^x n_q\delta^y{}_r \mp \delta^x{}_q n^y n_r + n^x n_q n^y n_r)\nabla_x V_y) - (a\leftrightarrow b).
\end{aligned}$$

Since h annihilates n, the only contributions can come from

$$\begin{aligned}
^{(d-1)}R_{abc}{}^d V_d &= h^p{}_a h^q{}_b h^r{}_c(\delta^x{}_q\delta^y{}_r\nabla_p\nabla_x V_y \mp n^x(\nabla_p n_q)\delta^y{}_r\nabla_x V_y \mp \delta^x{}_q n^y(\nabla_p n_r)\nabla_x V_y) - (a\leftrightarrow b)\\
&= h^p{}_a h^x{}_b h^y{}_c\nabla_p\nabla_x V_y \mp h^p{}_a h^q{}_b(\nabla_p n_q)h^y{}_c n^x\nabla_x V_y \mp h^p{}_a h^x{}_b h^r{}_c(\nabla_p n_r)n^y\nabla_x V_y - (a\leftrightarrow b)\\
&= h^p{}_a h^x{}_b h^y{}_c R_{pxyz}V^z \mp 2K_{[ab]}h^y{}_c n^x\nabla_x V_y \mp 2h^p{}_{[a}h^x{}_{b]}h^r{}_c(\nabla_p n_r)n^y\nabla_x V_y.
\end{aligned} \tag{8.20}$$

The second term is zero, for the third term use

$$n^y\nabla_x V_y = \nabla_x(n^y V_y) - V_y\nabla_x n^y = -V_y\nabla_x n^y, \tag{8.21}$$

so that

$$\begin{aligned}
{}^{(d-1)}R_{abc}{}^{d}V_d &= h^p{}_a h^x{}_b h^y{}_c R_{pxyz}V^z \pm 2h^p{}_{[a}h^x{}_{b]}h^r{}_c(\nabla_p n_r)V_y\nabla_x n^y \\
&= h^p{}_a h^x{}_b h^y{}_c R_{pxyz}V^z \pm 2K_{c[a}h^x{}_{b]}V_y\nabla_x n^y \\
&= h^p{}_a h^x{}_b h^y{}_c R_{pxyz}V^z \pm 2K_{c[a}h^x{}_{b]}\delta^y{}_z V^z\nabla_x n_y \\
&= h^p{}_a h^x{}_b h^y{}_c R_{pxyz}V^z \pm 2K_{c[a}h^x{}_{b]}(h^y{}_z \pm n^y n_z)V^z\nabla_x n_y \\
&= h^p{}_a h^x{}_b h^y{}_c R_{pxyz}V^z \pm 2K_{c[a}K_{b]z}V^z \\
&= \left(h^p{}_a h^x{}_b h^y{}_c R_{pxyz} \pm K_{ca}K_{bz} \mp K_{cb}K_{az}\right)V^z. \qquad (8.22)
\end{aligned}$$

We have obtained **Gauss' equation** Lect. 21

$$ {}^{(d-1)}R_{abcd} = h^p{}_a h^q{}_b h^r{}_c h^s{}_d R_{pqrs} \mp K_{ac}K_{bd} \pm K_{bc}K_{ad}. \qquad (8.23)$$

For the Ricci tensor, we obtain

$$ {}^{(d-1)}R_{bd} = h^{ac\,(d-1)}R_{abcd} = h^q{}_b h^s{}_d h^{pr}R_{pqrs} \mp KK_{bd} \pm K_{bc}K^c{}_d. \qquad (8.24)$$

You can also find a formula for the Ricci scalar:

$$\begin{aligned}
{}^{(d-1)}R &= h^{bd\,(d-1)}R_{bd} \\
&= h^{pr}h^{qs}R_{pqrs} \pm K^2 \pm K_{bc}K^{bc} \\
&= (g^{pr} \mp n^p n^r)(g^{qs} \mp n^q n^s)R_{pqrs} \mp K^2 \pm K_{bc}K^{bc} \\
&= R \mp 2n^p n^r R_{pr} \mp K^2 \pm K_{bc}K^{bc}. \qquad (8.25)
\end{aligned}$$

This is very useful if you want to divide up spacetime into space and time. There is another useful equation, the **Codazzi equation**. This comes from taking the divergence of K:

$$\begin{aligned}
{}^{(d-1)}\nabla_a K_c{}^a - {}^{(d-1)}\nabla_c K &= h^f{}_a h^e{}_c h^a{}_g \nabla_f K_e{}^g - h^b{}_c\nabla_b(h^{ad}\nabla_a n_d) \\
&= h^f{}_g h^e{}_c \nabla_f(h^g{}_x h^y{}_e \nabla_y n^x) - h^b{}_c\nabla_b(h^{ad}\nabla_a n_d) \\
&= h^f{}_x h^y{}_c \nabla_f\nabla_y n^x - h^b{}_c h^{ad}\nabla_b\nabla_a n_d \\
&= h^f{}_x h^y{}_c(\nabla_f\nabla_y n^x - \nabla_y\nabla_f n^x) \\
&= h^f{}_x h^y{}_c R_{fy}{}^x{}_z n^z = h^y{}_c R_{yz}n^z. \qquad (8.26)
\end{aligned}$$

We have used the fact that the connection ${}^{(d-1)}\nabla$ preserves h.
Two obvious uses for this formalism are the canonical formulation of general relativity (ADM formalism), and the positive energy theorem, which we will do next.

8.2 Spinors in Curved Spacetime

In Minkowski space, a spinor ψ is a four-component object which transforms in the following way: Under a Lorentz transformation

$$x \to x' = Lx, \qquad (8.27)$$

it transforms as

$$\psi \to \left(1 + \frac{1}{2}\gamma_{\mu\nu}\Lambda^{\mu\nu} + \dots\right)\psi, \qquad (8.28)$$

where

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{8.29}$$

and $\Lambda^{\mu\nu}$ are the parameters of an infinitesimal Lorentz transformation. The matrices $J_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu}$ are the generators of the Lorentz group in the spinor representation, the satisfy

$$[J_{\mu\nu}, J_{\rho\sigma}] = -\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho}. \tag{8.30}$$

What happens in an arbitrary spacetime? At each point, you can always construct the tangent space by finding the vierbeins satisfying $g_{ab} = e_a{}^\mu e_b{}^\nu \eta_{\mu\nu}$. Then under a local Lorentz transformation of the vierbeins, a spinor field transforms like a Minkowski space spinor:

$$\psi \to \left(1 + \frac{1}{2}\gamma_{\mu\nu}\Lambda^{\mu\nu}(x) + \dots\right)\psi. \tag{8.31}$$

Since you have a flat metric at each point, you define as before

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \cdot \mathbf{1}, \tag{8.32}$$

such that γ^0 is anti-Hermitian and γ^i are Hermitian.
In practice, a useful representation of the γ-matrices is

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{8.33}$$

where σ^i are the Pauli matrices. These matrices do not depend on the co-ordinates.
You can as always turn the Lorentz index into a spacetime index by contracting with e:

$$\gamma^a = e^a{}_\mu \gamma^\mu. \tag{8.34}$$

Then the matrices γ^a generally depend on co-ordinates.
You cannot accommodate spinors withour using either vielbeins or a basis of one-forms.

You want some notion of a **covariant derivative** of a spinor. This should be a quantity $D_a\psi$ which transforms as a spinor under local Lorentz transformations, and a covector under co-ordinate transformations.
It is easier to invent $D\psi$, a spinor-valued one-form, and treat ψ as a spinor-valued 0-form. This is

$$D\psi = \partial_a\psi \, dx^a + \frac{1}{4}\gamma_{\mu\nu}\omega^{\mu\nu}\psi. \tag{8.35}$$

As usual, under a Lorentz transformation of ψ, generated by Λ, the first term gives $\partial\Lambda$ terms, which are compensated by the connection. If ω is torsion-free, we have

$$dE^\mu = -\omega^\mu{}_\nu \wedge E^\nu, \tag{8.36}$$

and under a Lorentz transformation $E \to LE$ you get (schematically)

$$L\, dE + dL\, E = -\omega \wedge LE. \tag{8.37}$$

Adding a connection term cancels the $\partial\Lambda$ terms you get under a Lorentz transformation.

There is a **Ricci identity** for spinors:

$$\begin{aligned}
DD\psi &= \left(d+\frac{1}{4}\gamma_{\rho\sigma}\omega^{\rho\sigma}\wedge\right)\left(d\psi+\frac{1}{4}\gamma_{\mu\nu}\omega^{\mu\nu}\psi\right)\\
&= \frac{1}{4}\gamma_{\mu\nu}d\omega^{\mu\nu}\psi-\frac{1}{4}\gamma_{\mu\nu}\omega^{\mu\nu}d\psi+\frac{1}{4}\gamma_{\rho\sigma}\omega^{\rho\sigma}d\psi+\frac{1}{16}\gamma_{\rho\sigma}\gamma_{\mu\nu}\omega^{\rho\sigma}\wedge\omega^{\mu\nu}\psi\\
&= \left(\frac{1}{4}\gamma_{\mu\nu}d\omega^{\mu\nu}+\frac{1}{32}[\gamma_{\rho\sigma},\gamma_{\mu\nu}]\omega^{\rho\sigma}\wedge\omega^{\mu\nu}\right)\psi\\
&= \left(\frac{1}{4}\gamma_{\mu\nu}d\omega^{\mu\nu}+\frac{1}{16}\left(\eta_{\mu\rho}\gamma_{\nu\sigma}-\eta_{\nu\rho}\gamma_{\mu\sigma}-\eta_{\mu\sigma}\gamma_{\nu\rho}+\eta_{\nu\sigma}\gamma_{\mu\rho}\right)\omega^{\rho\sigma}\wedge\omega^{\mu\nu}\right)\psi\\
&= \left(\frac{1}{4}\gamma_{\mu\nu}d\omega^{\mu\nu}+\frac{1}{4}\gamma_{\nu\sigma}\omega_{\mu}{}^{\sigma}\wedge\omega^{\mu\nu}\right)\psi\\
&= \left(\frac{1}{4}\gamma_{\mu\nu}d\omega^{\mu\nu}+\frac{1}{4}\gamma_{\mu\nu}\omega^{\mu\tau}\wedge\omega_{\tau}{}^{\nu}\right)\psi=\frac{1}{4}\gamma_{\mu\nu}R^{\mu\nu}\psi,
\end{aligned}\tag{8.38}$$

where $R^{\mu\nu}$ is just the curvature 2-form. You could turn this into components:

$$(D_aD_b-D_bD_a)\psi=\frac{1}{4}R_{ab\mu\nu}\gamma^{\mu\nu}\psi=\frac{1}{4}R_{abcd}\gamma^{cd}\psi.\tag{8.39}$$

The Dirac equation is

$$(\gamma^aD_a+m)\psi=0.\tag{8.40}$$

An idea of great technological importance is that of a **constant spinor**:

$$D_a\psi=0,\tag{8.41}$$

these are 16 equations. If a constant spinor exists, one must have $D_aD_b\psi=0$ and hence

$$R_{abcd}\gamma^{cd}\psi=0.\tag{8.42}$$

Thus curvature is an obstruction to having a constant spinor. Lect.
We are trying to find constant spinors in Minkowski spacetime, i.e. solutions of 22

$$\nabla_a\epsilon=0.\tag{8.43}$$

We use the γ matrices that we defined before. This is best done in spherical co-ordinates, where the metric is

$$ds^2=-dt^2+dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\tag{8.44}$$

We pick a basis of one-forms:

$$E^0=dt,\quad E^1=dr,\quad E^2=r\,d\theta,\quad E^3=r\,\sin\theta\,d\phi\tag{8.45}$$

As usual we calculate the components of the connection one-form using

$$dE^\alpha=-\omega^\alpha{}_\beta\wedge E^\beta.\tag{8.46}$$

The nonvanishing connection components are

$$\omega^2{}_1 = \frac{1}{r}E^2, \quad \omega^3{}_2 = \frac{1}{r}\cot\theta E^3, \quad \omega^3{}_1 = \frac{1}{r}E^3. \tag{8.47}$$

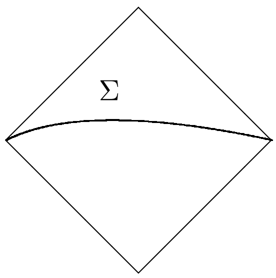
You find that the possible spinors are the following

$$\epsilon = \begin{pmatrix} e^{i\theta/2}(Ae^{i\phi/2} + Be^{-i\phi/2}) \\ e^{-i\theta/2}(Ae^{i\phi/2} - Be^{-i\phi/2}) \\ e^{i\theta/2}(Ce^{i\phi/2} + De^{-i\phi/2}) \\ e^{-i\theta/2}(Ce^{i\phi/2} - De^{-i\phi/2}) \end{pmatrix}, \tag{8.48}$$

where A, B, C and D are complex constants.
There is something inherently spinorial about this. If you rotate $\phi \mapsto \phi + 2\pi$ around the z-axis, ϵ will go to $-\epsilon$. That is characteristic of a spinor which is not single-valued in spacetime.
Under a rotation $\phi \mapsto \phi + 4\pi$, the spinor transforms into itself. This reflects the fact that spinors are *not* representations of $SO(3,1)$, but rather its universal cover.

8.3 Definition of Mass

In general relativity, conserved quantities are only associated with Killing vectors. It is difficult to give a definition of energy.
We consider asymptotically flat spacetimes, which have Penrose diagram



Here Σ is some spacelike surface.
You would like to invent a Gaussian integral; we need to find a two-form to integrate over the S^2 at infinity.
For stationary spacetimes there is a notion of energy, since we have a Killing vector $k^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t}$.
k then defines a one-form and you can integrate

$$\frac{1}{8\pi}\int_{S^2_\infty} *dk = M \text{ for Schwarzschild.} \tag{8.49}$$

All that is required for this definition is a metric asymptotic to the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r} + \dots\right)dt^2 + \left(1 - \frac{2M}{r} + \dots\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{8.50}$$

The one-form k associated with the Killing vector is

$$k = \left(-1 + \frac{2M}{r} + \dots\right)dt, \tag{8.51}$$

then

$$dk = \left(-\frac{2M}{r^2} + \ldots\right) dr \wedge dt = \frac{2M}{r^2} dt \wedge dr + \ldots,$$
$$*dk = \frac{2M}{r^2} r^2 \sin\theta \, d\theta \wedge d\phi + \ldots = 2M \sin\theta \, d\theta \wedge d\phi + O\left(\frac{1}{r}\right). \tag{8.52}$$

Hence

$$\int_{S^2_\infty} *dk = \int_{S^2_\infty} 2M \sin\theta \, d\theta \wedge d\phi = 8\pi M, \tag{8.53}$$

which works for any stationary metric.
We need to apply the divergence theorem to this:

$$M = -\frac{1}{4\pi} \int_{S^2_\infty} dS^{ab} \, \nabla_a k_b = -\frac{1}{4\pi} \int_\Sigma d\Sigma^b \, \nabla^a \nabla_a k_b, \tag{8.54}$$

where Σ is any surface asymptotic to a two-sphere. Since

$$\Box k_b = -\nabla^a \nabla_b k_a = -\nabla_b \nabla^a k_a - R^a{}_{bac} k^c = -R_{bc} k^c, \tag{8.55}$$

where we have used Killing's equation $\nabla_{(a} k_{b)} = 0$, this becomes

$$M = \frac{1}{4\pi} \int_\Sigma d\Sigma^b \, R_{bc} k^c. \tag{8.56}$$

You can pick the surface such that k is everywhere normal to Σ. Then

$$M = \frac{1}{4\pi} \int_\Sigma d\Sigma \, R_{bc} k^b k^c = 2 \int_\Sigma d\Sigma \left(T_{ab} - \frac{1}{2} T g_{ab}\right) k^a k^b \tag{8.57}$$

by the Einstein equations. This is the closest you can get to something like a Gaussian integral in general relativity. For a perfect fluid,

$$T_{ab} = (p + \rho) u_a u_b + p g_{ab}, \tag{8.58}$$

where ρ is the energy density of the fluid, p its pressure and u its velocity. If you pick an orthonormal frame, where $u^\mu = (1, 0, 0, 0)$,

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad T = 3p - \rho. \tag{8.59}$$

For $k^\mu = (1, 0, 0, 0)$, the mass is

$$M = 2 \int_\Sigma d\Sigma \left(\rho + \frac{1}{2}(3p - \rho)\right) = \int_\Sigma d\Sigma (\rho + 3p). \tag{8.60}$$

Why should M be positive? If not, perpetual motion machines using gravity seem possible. This would be a sign of instability in the theory.
In Newtonian theory (+ special relativity), the total energy would be

$$\text{rest mass energy } + \text{ kinetic energy } + \text{ potential energy.} \tag{8.61}$$

Since potential energy is negative and scales with M^2, there is no guarantee that this is bounded by the positive contributions.
In general relativity, an example of a spacetime with negative energy is given by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{8.62}$$

M is just a constant of integration, and the spacetime with negative M is still a solution of $R_{ab} = 0$ locally. It contains a naked singularity at $r = 0$.

8.4 Energy Conditions

We must require the energy-momentum tensor to satisfy certain conditions. Possible conditions are

- **Weak Energy Condition**

  $$T^{ab} t_a t_b \geq 0 \tag{8.63}$$

  for any timelike vector t. This means the energy density must be positive in any frame. In an orthonormal frame in which the matter is at rest,

  $$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{8.64}$$

  is isotropic in space. We can therefore take t to be

  $$t_\mu = (\cosh\theta, \sinh\theta, 0, 0). \tag{8.65}$$

  Then we have

  $$\rho \cosh^2\theta + p \sinh^2\theta \geq 0. \tag{8.66}$$

  Setting $\theta = 0$, we see we must have $\rho \geq 0$; in the limit $\theta \to \infty$ we get $\rho \geq -p$. Almost all known forms of matter obey this condition. It is not very useful for proving theorems.

- **Dominant Energy Condition**
  This states that $T^{00} \geq |T^{ab}|$ in any orthonormal frame, or

  $$\rho \geq |p| \geq 0. \tag{8.67}$$

  Another way of expressing this is to say that

  $$w^a = T^{ab} t_b \tag{8.68}$$

  is not spacelike for arbitrary timelike or null vectors t_b. Examples of use are

  (i) If the dominant energy condition holds, the event horizon of a black hole is spherical in $d = 4$.

(ii) If the dominant energy condition holds, energy in general relativity is positive (see below.)

Examples:

1. The condition holds for all plausible classical matter including a cosmological constant.
2. The condition is not true in QFT. Consider the Casimir effect which has negative energy density.

- **Strong Energy Condition** Lect.
This is useful for singularity theorems. The condition is 23

$$R_{ab}t^a t^b \geq 0, \tag{8.69}$$

where t^a is an arbitrary timelike or null vector, and is a geometrical condition rather than a physical one. You can translate this into a condition on T_{ab} via the Einstein equations:

$$\left(T_{ab} - \frac{1}{2}T g_{ab}\right) t^a t^b \geq 0. \tag{8.70}$$

If you pick t to be a unit vector as before, then you can turn this into conditions on the pressure and energy density:

$$p + \rho \geq 0, \quad 3p + \rho \geq 0. \tag{8.71}$$

This proves positivity of mass for static spacetimes, where k^a is everywhere timelike, and

$$M = \int d\Sigma\,(\rho + 3p). \tag{8.72}$$

The strong energy condition does not hold for a positive cosmological constant, which has

$$\rho > 0, \quad p = -\rho. \tag{8.73}$$

This messes up some singularity theorems.

8.5 Proof of Positive Energy

We want to prove that energy is positive in general. We suppose our spacetime is asymptotically flat, and contains a spacelike surface Σ that does not contain singularities. This will have an induced metric h_{ab} and an outward normal t^a, which we assume is normalised. We assume the surface is asymptotic to a two-surface with volume element

$$dS^{ab} = \frac{1}{2}(t^a r^b - t^b r^a) dS. \tag{8.74}$$

Asymptotically, we have $t^a \sim (1,0,0,0)$ and $r^a \sim (1,0,0,0)$ (in spherical polar co-ordinates). Then

$$dS^{01} \sim \frac{1}{2} r^2 \sin\theta \, d\theta \, d\phi, \quad dS^{10} \sim -\frac{1}{2} r^2 \sin\theta \, d\theta \, d\phi, \quad \text{all others vanish.} \tag{8.75}$$

All that is needed is to find a vector field X^a such that $X \sim \frac{m}{r^2}$ in the r-direction. Then

$$I = -\frac{1}{8\pi} \int_{S^2_\infty} dS^{ab}\, t_a X_b \tag{8.76}$$

will correspond to mass. The idea is to turn this into an integral over Σ and find something that is positive, applying the divergence theorem:

$$\begin{aligned} I &= -\frac{1}{16\pi} \int_{S^2_\infty} dS^{ab}\,(t_a X_b - t_b X_a) \\ &= -\frac{1}{16\pi} \int_\Sigma d^3x\, \sqrt{h} t^a\, \nabla^b (t_a X_b - t_b X_a) \\ &= -\frac{1}{16\pi} \int_\Sigma d^3x\, \sqrt{h} \left((t^a \nabla^b t_a) X_b + t^a t_a \nabla^b X_b - (t^a \nabla^b t_b) X_a - t^a t_b \nabla^b X_a \right) \\ &= -\frac{1}{16\pi} \int_\Sigma d^3x\, \sqrt{h} \left(-\nabla^b X_b - (t^a \nabla^b t_b) X_a - t^a t_b \nabla^b X_a \right), \end{aligned} \tag{8.77}$$

where we write $d\Sigma^a = d^3x\, \sqrt{h}\, t^a$ and we have used $t^a t_a = -1$. Since $h^{ab} = g^{ab} + t^a t^b$, we can write this as

$$I = \frac{1}{16\pi} \int_\Sigma d^3x\, \sqrt{h} \left(h^{ab} \nabla_a X_b + t^a X_a \nabla^b t_b \right). \tag{8.78}$$

If we choose a vector field X that lies in Σ everywhere, the second term vanishes:

$$I = \frac{1}{16\pi} \int_\Sigma d^3x\, \sqrt{h} \left(h^{ab} \nabla_a X_b \right). \tag{8.79}$$

To show that this is positive, you have to invent something that turns this into an integral over a square of soemthing. The only possibility is to use a spinor to do this:

$$X_a = \epsilon^\dagger \nabla_a \epsilon. \tag{8.80}$$

Then

$$h^{ab} \nabla_a X_b = h^{ab} \nabla_a \epsilon^\dagger \nabla_b \epsilon + h^{ab} \epsilon^\dagger \nabla_a \nabla_b \epsilon \tag{8.81}$$

To get an X which actually lies in Σ, we project it:

$$X^a = h^{ab} \epsilon^\dagger \nabla_b \epsilon. \tag{8.82}$$

The motivation to do things this way came from supergravity.
We must invent an equation for ϵ to satisfy. Remember we need $X^r \sim \frac{m}{r^2}$ near infinity. A first guess would be $\epsilon \sim r^{-1/2}$, but does not quite work. We rather assume boundary conditions

$$\epsilon \to \text{constant as } r \to \infty. \tag{8.83}$$

It is possible to find such an ϵ near infinity.
We only need a spinor in the surface Σ. Consider the Dirac equation $\gamma^a \nabla_a \epsilon = 0$ which does not only describe things in Σ, so take its projection into Σ. This is the **Witten equation**

$$h^{ab} \gamma_a \nabla_b \epsilon = 0. \tag{8.84}$$

The rest of this is an awful calculation. We write

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix}. \tag{8.85}$$

You can pick one of the spinors that are covariantly constant in flat space:

$$\epsilon_1 = e^{i(\theta+\phi)/2}, \quad \epsilon_2 = e^{i(\phi-\theta)/2}, \quad \epsilon_3 = \epsilon_4 = 0. \tag{8.86}$$

The metric on Σ, near infinity, must look like the Schwarzschild metric,

$$ds^2_\Sigma = \left(1 + \frac{2M}{r} + \dots\right) dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{8.87}$$

If you solve the Witten equation in powers of $\frac{1}{r}$, you find

$$\epsilon_1 = e^{i(\theta+\phi)/2}\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right), \quad \epsilon_2 = e^{i(\phi-\theta)/2}\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right), \quad \epsilon_3 = \epsilon_4 = 0. \tag{8.88}$$

Then near infinity, X^r is

$$X^r = h^r{}_b \epsilon^\dagger \nabla^b \epsilon = e^{-i(\theta+\phi)/2} \frac{2M}{r^2} e^{i(\theta+\phi)/2} + \frac{2M}{r^2} + \dots = \frac{4M}{r^2} + \dots \tag{8.89}$$

Then indeed

$$M = -\frac{1}{16\pi} \int_{S^2_\infty} dS_{ab}\, t^a X^b, \tag{8.90}$$

so we can take the integral I as a defintion of mass.
How do you know that such a solution to the Witten equation exists? Let us write

$$W = h^{ab} \gamma_a \nabla_b, \tag{8.91}$$

so that we try to find a solution of $W\epsilon = 0$. You can write $\epsilon = \epsilon_0 + \epsilon_1$, where $\nabla_a \epsilon_0 = 0$ in flat space. Then we want

$$W\epsilon_1 = -W\epsilon_0, \tag{8.92}$$

where the right-hand side is fixed. You can find the Green's function of W, call it G. Then

$$\epsilon_1(x) = -\int d^3x\, \sqrt{h}\, G(x, x')(W\epsilon_0(x')). \tag{8.93}$$

Since this method seems to be able to prove existence of any kind of solution you can think of, it is not a rigorous proof, which however apparently exists. Now let us go back to

$$M = \frac{1}{16\pi} \int_\Sigma d^3x\, \sqrt{h}\, h^{ab} \nabla_a X_b, \quad X_b = h^a{}_b \epsilon^\dagger \nabla_a \epsilon. \tag{8.94}$$

Square the Witten equation to get

$$\begin{aligned}
0 &= h^{cd}\gamma_c\nabla_d\left(h^{ab}\gamma_a\nabla_b\epsilon\right)\\
&= h^{cd}\gamma_c h^{ab}\gamma_a\nabla_d\nabla_b\epsilon + h^{cd}\gamma_c\gamma_a(\nabla_b\epsilon)(\nabla_d h^{ab})\\
&= h^{cd}h^{ab}\left(g_{ca}+\gamma_{ca}\right)\nabla_d\nabla_b\epsilon + h^{cd}\gamma_c\gamma_a(\nabla_b\epsilon)(\nabla_d h^{ab})\\
&= h^{db}\nabla_d\nabla_b\epsilon + h^{cd}h^{ab}\gamma_{ca}\nabla_{[d}\nabla_{b]}\epsilon + h^{cd}\gamma_c\gamma_a(\nabla_b\epsilon)(\nabla_d h^{ab})\\
&= h^{db}\nabla_d\nabla_b\epsilon + \frac{1}{8}h^{cd}h^{ab}\gamma_{ca}R_{dbef}\gamma^{ef}\epsilon + h^{cd}\gamma_c\gamma_a(\nabla_b\epsilon)(\nabla_d h^{ab}),
\end{aligned} \tag{8.95}$$

where we have used the Ricci identity. The first term is a sort of Laplacian projected into Σ. We will use this to show that the mass integral is positive: Lect. 24

$$\begin{aligned}
M &= \frac{1}{16\pi}\int_\Sigma d^3x\,\sqrt{h}\,h^{ab}\nabla_a\left(h^c{}_b\epsilon^\dagger\nabla_c\epsilon\right)\\
&= \frac{1}{16\pi}\int_\Sigma d^3x\,\sqrt{h}\left(h^{ab}(\nabla_a\epsilon^\dagger)h^c{}_b\nabla_c\epsilon + h^{ab}h^c{}_b\epsilon^\dagger\nabla_a\nabla_c\epsilon + h^{ab}\left(\nabla_a h^c{}_b\right)\epsilon^\dagger\nabla_c\epsilon\right)
\end{aligned} \tag{8.96}$$

Since the Dirac conjugate in a curved spacetime must be taken to be

$$\bar{\epsilon} = \epsilon^\dagger\gamma^a t_a, \tag{8.97}$$

we have

$$\bar{\epsilon}\gamma^b t_b = \epsilon^\dagger\gamma^a\gamma_b t_a t_b = \epsilon^\dagger t^a t_a = -\epsilon^\dagger. \tag{8.98}$$

Hence

$$(\nabla_a\epsilon)^\dagger = -\overline{\nabla_a\epsilon}\gamma^b t_b = -\nabla_a\bar{\epsilon}\gamma^b t_b = -\nabla_a(\epsilon^\dagger\gamma^c t_c)\gamma^b t_b = \nabla_a\epsilon^\dagger - \epsilon^\dagger\gamma^c(\nabla_a t_c)\gamma^b t_b, \tag{8.99}$$

and

$$M = \frac{1}{16\pi}\int_\Sigma d^3x\,\sqrt{h}\left(h^{ac}(\nabla_a\epsilon)^\dagger\nabla_c\epsilon + h^{ac}\epsilon^\dagger\gamma^d(\nabla_a t_d)\gamma^b t_b\nabla_c\epsilon + h^{ac}\epsilon^\dagger\nabla_a\nabla_c\epsilon + h^{ab}\left(\nabla_a h^c{}_b\right)\epsilon^\dagger\nabla_c\epsilon\right). \tag{8.100}$$

The first term is positive, and only zero if $\nabla_a\epsilon = 0$ everywhere. For the remaining terms, use the squared Witten equation to get

$$\begin{aligned}
M = \frac{1}{16\pi}\int_\Sigma d^3x\,\sqrt{h}\Big(&h^{ac}(\nabla_a\epsilon)^\dagger\nabla_c\epsilon + h^{ac}\epsilon^\dagger\gamma^d(\nabla_a t_d)\gamma^b t_b\nabla_c\epsilon + h^{ab}\left(\nabla_a h^c{}_b\right)\epsilon^\dagger\nabla_c\epsilon\\
&-\frac{1}{8}\epsilon^\dagger h^{cd}h^{ab}\gamma_{ca}R_{dbef}\gamma^{ef}\epsilon - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)(\nabla_d h^{ab})\Big).
\end{aligned} \tag{8.101}$$

First consider the term involving the Riemann tensor:

$$\begin{aligned}
-\frac{1}{8}\epsilon^\dagger h^{cd}h^{ab}\gamma_{ca}R_{dbef}\gamma^{ef}\epsilon &= -\frac{1}{8}\epsilon^\dagger\left(g^{cd}+t^c t^d\right)\left(g^{ab}+t^a t^b\right)\gamma_c\gamma_a\gamma_e\gamma_f R_{db}{}^{ef}\epsilon\\
&= -\frac{1}{8}\epsilon^\dagger\left(R^{caef}\gamma_c\gamma_a\gamma_e\gamma_f + 2t^c t^d R_d{}^{aef}\gamma_c\gamma_a\gamma_e\gamma_f\right)\epsilon.
\end{aligned} \tag{8.102}$$

From the Bianchi identity,

$$\begin{aligned}
R_d{}^{aef}\gamma_a\gamma_e\gamma_f &= -\left(R_d{}^{efa}+R_d{}^{fae}\right)\gamma_a\gamma_e\gamma_f\\
&= -R_d{}^{efa}\left(\gamma_e\gamma_f\gamma_a + 2g_{ae}\gamma_f - 2g_{af}\gamma_e\right) - R_d{}^{fae}\left(\gamma_f\gamma_a\gamma_e + 2g_{ef}\gamma_a - 2g_{af}\gamma_e\right)\\
&= -2R_d{}^{aef}\gamma_a\gamma_e\gamma_f - 6R_d{}^f\gamma_f,
\end{aligned} \tag{8.103}$$

hence

$$R_d{}^{aef}\gamma_a\gamma_e\gamma_f = -2R_d{}^f\gamma_f \tag{8.104}$$

and

$$\begin{aligned}
-\frac{1}{8}\epsilon^\dagger h^{cd}h^{ab}\gamma_{ca}R_{dbef}\gamma^{ef}\epsilon &= -\frac{1}{8}\epsilon^\dagger\left(-2R - 4t^ct^dR_d{}^f\gamma_c\gamma_f\right)\epsilon \\
&= \frac{1}{8}\epsilon^\dagger\left(2R + 4t^ct^d\left(8\pi T_d{}^f + \frac{1}{2}R\delta_d{}^f\right)\gamma_c\gamma_f\right)\epsilon \\
&= 4\pi T_d{}^f\left(\epsilon^\dagger t^c\gamma_c\gamma_f\epsilon\right)t^d,
\end{aligned} \tag{8.105}$$

where we used the Einstein equations. Now

$$w_f := \epsilon^\dagger t^c\gamma_c\gamma_f\epsilon \tag{8.106}$$

is a timelike vector (check), and so if dominant energy is satisfied, this term is positive. There are three other terms in M which cancel:

$$\begin{aligned}
& h^{ac}\epsilon^\dagger\gamma^d(\nabla_a t_d)\gamma^b t_b\nabla_c\epsilon + h^{ab}\left(\nabla_a h^c{}_b\right)\epsilon^\dagger\nabla_c\epsilon - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)(\nabla_d h^{ab}) \\
= \;& h^{ac}\epsilon^\dagger\gamma^d(\nabla_a t_d)\gamma^b t_b\nabla_c\epsilon + h^{ab}\nabla_a\left(t^c t_b\right)\epsilon^\dagger\nabla_c\epsilon - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)\nabla_d\left(t^a t^b\right) \\
= \;& h^{ac}\epsilon^\dagger\gamma^d(\nabla_a t_d)\gamma^b t_b\nabla_c\epsilon + h^{ab}t^c\left(\nabla_a t_b\right)\epsilon^\dagger\nabla_c\epsilon - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)t^a\left(\nabla_d t^b\right) - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)t^b\left(\nabla_d t^a\right) \\
= \;& h^{ac}\epsilon^\dagger\gamma^d K_{ad}\gamma^b t_b\nabla_c\epsilon + h^{ab}t^c K_{ab}\epsilon^\dagger\nabla_c\epsilon - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)t^a K_d{}^b - h^{cd}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)t^b K_d{}^a \\
= \;& \epsilon^\dagger\gamma^d K^c{}_d\gamma^b t_b\nabla_c\epsilon + t^c K\epsilon^\dagger\nabla_c\epsilon - \epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)t^a K^{cb} - K^{ca}\epsilon^\dagger\gamma_c\gamma_a(\nabla_b\epsilon)t^b = 0.
\end{aligned} \tag{8.107}$$

One is left with

$$M = \frac{1}{16\pi}\int_\Sigma d^3x\,\sqrt{h}\left(h^{ac}(\nabla_a\epsilon)^\dagger\nabla_c\epsilon + 4\pi T_d{}^f w^f t^d\right). \tag{8.108}$$

You can interpret the first part as the energy of the gravitational field, and the second as the energy of matter. The amazing thing is the way is proof is done is based on supergravity.

The result that $M \geq 0$ as long as dominant energy holds is absolutely true in classical general relativity.

It must mean that gravitational energy is not localised: Imagine some matter distribution in a region on the surface Σ, then you could deform Σ slightly to a new surface Σ' not including this region. M would be the same for Σ and Σ', but the contributions from the two terms could be quite different.

The partition between "gravitational" energy and "matter" energy is different on different surfaces. So one sees that gravitational energy cannot be localised.

Finally,

$$M = 0 \quad \Rightarrow \nabla_a\epsilon = 0, \quad T_{ab} = 0, \tag{8.109}$$

so if spacetime is asymptotically flat with no horizons and zero mass, it must be flat space.

- END -