Solving matrix equation $A_2X^2 + A_1X + A_0 = 0$

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We are interested in real solutions to equation

$$A_2X^2 + A_1X + A_0 = 0$$

where $A_i, \quad i = 0, 1, 2$ are square matrices of order n and we want to know if there exist a unique solution where the eigenvalues of the solution X are inside the unit circle.

We consider two algorithms: cyclic reduction and generalized Schur decomposition.

We can take advantage of the zero columns of $A_i, \quad i = 0, 1, 2$. Let's define ι_i as the set of indices of non-zero columns of matrix A_i and ι_b, the intersection of ι_0 and ι_2, corresponding to the column that aren't zero in A_0 and in A_2. The set $\iota_m = \iota_0 \setminus \iota_b$ indicates the columns that are non-zero in A_0 but not in A_2. b_{ι_0} collects the indices of the element of b in ι_0 and b_{ι_2} collects the indices of the element of b in ι_2.

We note M^c a matrix made of a selection of its columns in set c and $M^{(r,c)}$ a submatrix made of rows in set r and columns in set c.

1 Generalized Schur decomposition

Before using generalized Schur decomposition, we need to transform the original problem in a linear one. If

$$A_2X^2 + A_1X + A_0 = 0$$

we can rewrite the equation as

$$\begin{bmatrix} A_1 & A_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X = \begin{bmatrix} -A_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix}$$

or, taking into account the empty columns in $A_i, \quad i = 0, 1, 2$

$$\begin{bmatrix} A_1^{(\iota_m)} & A_2^{(\iota_2)} \\ I^{(b_{\iota_m})} & 0 \end{bmatrix} \begin{bmatrix} I \\ X^{(\iota_2,\iota_0)} \end{bmatrix} X^{(\iota_0,\iota_0)} = \begin{bmatrix} -A_0^{(\iota_0)} & -A_1^{(\iota_2)} \\ 0 & I^{(b_{\iota_2})} \end{bmatrix} \begin{bmatrix} I \\ X^{(\iota_2,\iota_0)} \end{bmatrix}$$

where I matrices are conformant. $I^{(b_{\iota_i})}$ is a matrix with as many columns as $A_i^{(\iota_i}$ and made of unit vectors in columns b_{ι_i} for $i = 0, 2$. Note that the choice of putting columns $A_1^{\iota_b}$ on the right hand side is arbitrary but has no implication on the results.

Let's note

$$D = \begin{bmatrix} A_1^{(\iota_m)} & A_2^{(\iota_2)} \\ I^{(b_{\iota_m})} & 0 \end{bmatrix}$$

and

$$E = \begin{bmatrix} -A_0^{(\iota_0)} & -A_1^{(\iota_2)} \\ 0 & I^{(b_{\iota_2})} \end{bmatrix}$$

Then we can use the generalized Schur decomposition to select an X matrix with all eigenvalues inside the unit circle and check that it is unique.